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December 2016 | ₹ 35.00

MATHEMATICS Spectrum

TARGET 2017

Rapid

Concept Revision

Inequalities
Logarithms
Quadratic Equation
Sets, Relations & Functions
Mathematical Reasoning
Complex Numbers

FROM CLASS 11 SYLLABUS

Matrix
Determinant
Function
Limits
Continuity & Differentiability
Differentiation

FROM CLASS 12 SYLLABUS

TEST RIDER FOR JEE
TEST DRIVE FOR BOARD

JUVENILEZ

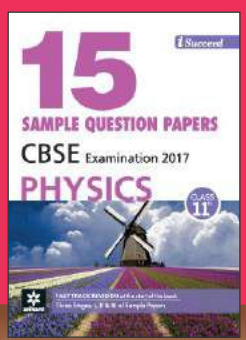
GOLDEN OLDIES *Trigonometry*
RAINBOW *Sequences and Series*
FUN ARCADE

MEGA ISSUE 1
More Pages
More Features
More Practice...

15 SAMPLE QUESTION PAPERS

CBSE CLASS XI & XII
EXAMINATION 2017

Fast Track Revision at the Start of the Book Three Stages; I,II & III of Sample Papers
Latest CBSE Sample Paper & Solved Paper in Class XII



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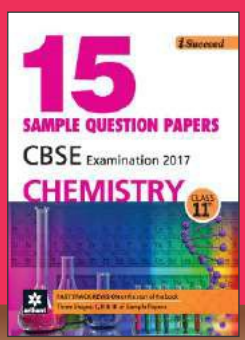

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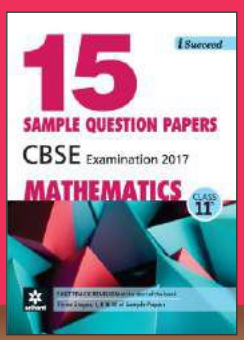

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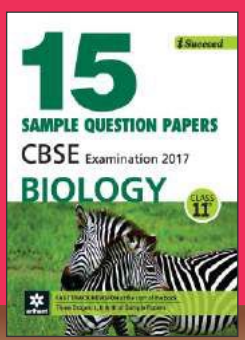

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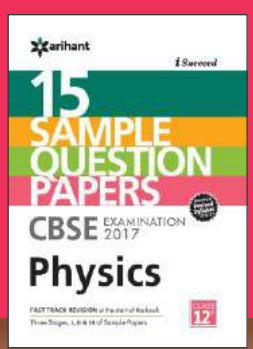

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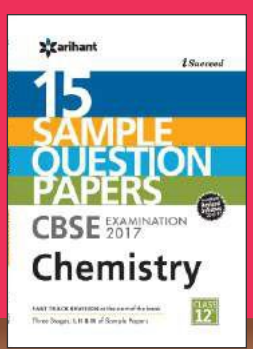

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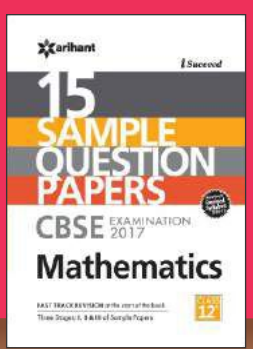

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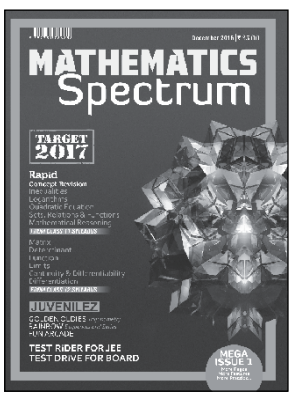


Dear Aspirants,

For many of you the month of December is very precious as it is the first month to gear up your studies. For an aspirant it is very difficult to maintain a proper balance between emotions, performance, attitude, work pressure, and time availability during this month. This balancing requires cool, calm and focused approach. So, in my opinion one must follow "no aggression - only devotion" policy during this month.

Change or Updations are the prime requirements to live in a system with dynamically changing environment. We as Spectrum family also follow the same, and always try to incorporate something new for you. With this ideology in this first Mega issue of this season with ***more pages, more practice and more features*** we are introducing few new columns from this issue, i.e.

Rapid Concept Revision
Test Drive and
Rainbow

All of these columns are targeted to fine tune your preparations in synchrony with Target 2017. Hope, these columns will prove their worth with a difference and you will like the change. Remember

Challenges are what make life interesting. Overcoming them is what makes life meaningful ***Joshua J. Marine***

So, challenge yourself to succeed. The spectrum team will be anxiously waiting for your feedbacks and comments.

Good luck

Contents

Rapid Concept Revision @ Class XI (02)
Test Rider JEE Main Prep Up (29)
Test Rider JEE Main Scale Up (35)
Golden Oldies (37)
JEE Master Bash (42)
Concept Map (48)
Rapid Concept Revision @ Class XII (50)
Rainbow (74)
JEE Advanced Drill (76)
Brain Teasers (83)
Test Drive Board Exam Prep Up (85)
Test Drive Board Exam Scale Up (91)
Fun Arcade (94)
Quizzer (96)

@ CLASS XI SYLLABUS

Rapid CONCEPT REVISION

TARGET 2017

INEQUALITIES

A statement involving variable(s) and the sign of inequality *viz.*, $>$, $<$, $\geq$ and $\leq$ is called an inequation.

Elementry Properties of Inequalities

The following are some very useful points to remember

(i) $a \leq b$ either $a < b$ or $a = b$

(ii) $a < b$ and $b < c \Rightarrow a < c$ (transition property)

(iii) $a < b \Rightarrow -a > -b$, i.e. inequality sign reverses, if both sides are multiplied by a negative number.

(iv) $a < b$ and $c < d \Rightarrow a + c < b + d$ and $a - d < b - c$.

(v) If both sides of inequality are multiplied (or divided) by a positive number, inequality does not change. When both of its sides are multiplied (or divided) by a negative number, inequality gets reserved. Thus, $a < b \Rightarrow ka < kb$, if $k > 0$ and $ka > kb$, if $k < 0$.

(vi) $0 < a < b \Rightarrow a^r < b^r$, if $r > 0$ and $a^r > b^r$, if $r < 0$.

(vii) $a + \frac{1}{a} \geq 2$ for $a > 0$ and equality holds for $a = 1$.

(viii) $a + \frac{1}{a} \leq -2$ for $a < 0$ and equality holds for $a = -1$.

(ix) **Squaring an inequality** If $a < b$, then $a^2 < b^2$ does not follows always. In fact $a < b \Rightarrow a^2 < b^2$ follows only when the absolute value of a is less than the absolute value of the b or distance of a from zero is less than the distance of b from zero.

(x) **Law of reciprocal** If both sides of inequality have same sign, while taking its reciprocal the sign of inequality gets reversed. Thus,

$$a > b > 0 \Rightarrow \frac{1}{a} < \frac{1}{b} \text{ and } a < b < 0 \Rightarrow \frac{1}{a} > \frac{1}{b}$$

But if both sides of inequality have opposite sign, then while taking its reciprocal the sign of inequality does not change.

Thus, $a < 0 < b \Rightarrow \frac{1}{a} < \frac{1}{b}$

If $x \in [a, b] \Rightarrow$

$$\begin{cases} \frac{1}{x} \in \left[\frac{1}{b}, \frac{1}{a}\right], & \text{if } a \text{ and } b \text{ have same sign} \\ \frac{1}{x} \in \left(-\infty, \frac{1}{a}\right] \cup \left[\frac{1}{b}, \infty\right), & \text{if } a \text{ and } b \text{ have opposite signs} \end{cases}$$

Generalized Method of Intervals For Solving Inequalities

Let $F(x) = (x - a_1)^{k_1} (x - a_2)^{k_2} \ldots (x - a_{n-1})^{k_{n-1}} (x - a_n)^{k_n}$

where $k_1, k_2, \ldots, k_n \in Z$ and $a_1, a_2, \ldots, a_n$ are fixed real numbers satisfying the condition $a_1 < a_2 < a_3 < \ldots < a_{n-1} < a_n$

For solving $F(x) > 0$ or $F(x) < 0$, consider the following algorithm

(i) We mark the numbers $a_1, a_2, \ldots, a_n$ on the number axis and put a plus sign in the interval on the right of the largest of these numbers, i.e. on the right of a_n.

(ii) Then, we put the plus sign in the interval on the left of a_n, if k_n is an even number and a minus sign, if k_n is an odd number. In the next interval, we put a sign according to the following rule.

When passing through the point a_{n-1}, the polynomial $F(x)$ changes sign, if k_{n-1} is an odd number. Then, we consider the next interval and put a sign in it using the same rule.

(iii) Thus, we consider all the intervals. The solution of the inequality $F(x) > 0$ is the union of all intervals in which we have put the plus sign and the solution of the inequality $F(x) < 0$ is the union of all intervals in which we have put the minus sign.

Some Important Inequality

(i) $(x-a)(x-b)<0$
$\Rightarrow x \in (a, b)$, where $a<b$

(ii) $(x-a)(x-b)>0$
$\Rightarrow x \in (-\infty, a) \cup (b, \infty)$, where $a<b$

(iii) $x^2 \le a^2 \Rightarrow x \in [-a, a]$

(iv) $x^2 \ge a^2 \Rightarrow x \in (-\infty, -a] \cup [a, \infty)$

(v) If $ax^2+bx+c<0, (a>0)$
$\Rightarrow x \in (\alpha, \beta)$, where $\alpha, \beta (\alpha<\beta)$ are roots of the equation $ax^2+bx+c=0$.

(vi) If $ax^2+bx+c>0, (a>0) \Rightarrow x \in (-\infty, \alpha) \cup (\beta, \infty)$, where $\alpha, \beta (\alpha<\beta)$ are roots of the equation $ax^2+bx+c=0$.

Inequalities Involving Absolute Value

(i) If $|x| \le a$ (where $a>0$), then $x \in [-a, a]$.

(ii) If $|x| \ge a$ (where $a>0$), then $x \in (-\infty, -a] \cup [a, \infty)$.

(iii) If $a \le |x| \le b$ (where $a, b>0$), then $x \in [-b, -a] \cup [a, b]$.

(iv) If $|x-a| \le r$, then $x \in [a-r, a+r]$.

(v) If $|x-a| \ge r$, then $x \in (-\infty, a-r] \cup [a+r, \infty)$.

(vi) $|x+y|<|x|+|y|$, if x and y have opposite signs.

(vii) $|x-y|<|x|+|y|$, if x and y have same sign.

(viii) $|x+y|=|x|+|y|$, if x and y have same sign or atleast one of x and y is zero.

(ix) $|x-y|=|x|+|y|$, if x and y have opposite signs or atleast one of x and y is zero.

MASTER STROKES

1. The number of real solutions of the equation $|x|^2-3|x|+2=0$ is
(a) 4 *(b)* 1 *(c)* 2 *(d)* 3

2. The largest interval for which $x^{12}-x^9+x^4-x+1>0$ is
(a) $-4<x\le 0$ *(b)* $0<x<1$
(c) $-100<x<100$ *(d)* $-\infty<x<\infty$

3. If x satisfies $x^2-|x+2|+x>0$, then
(a) $(-\infty, 2)$ *(b)* $(-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$
(c) $(-\infty, -1) \cup (1, \infty)$ *(d)* $(\sqrt{2}, \infty)$

4. If x satisfies $|x-1|+|x-2|+|x-3| \ge 6$, then
(a) $0 \le x \le 4$ *(b)* $x \le -2$ or $x \ge 4$
(c) $x \le 0$ or $x \ge 4$ *(d)* None of these

5. The number of integral solutions of $\frac{x+2}{x^2+1} > \frac{1}{2}$ is
(a) 4 *(b)* 5 *(c)* 3 *(d)* 2

6. If $\left|\frac{x^2+6}{5x}\right| \ge 1$, then x belongs to
(a) $(-\infty, -3)$
(b) $(-\infty, -3) \cup (3, \infty)$
(c) $(-\infty, -3] \cup [-2, 0) \cup (0, 2] \cup [3, \infty)$
(d) R

7. The solution set of the inequation $|x|-1<1-x$, is
(a) $(-1, 1)$ *(b)* $(0, \infty)$
(c) $(-1, \infty)$ *(d)* None of these

8. The solution set of the inequation $|2x-3|<|x+2|$, is
(a) $(-\infty, 1/3)$ *(b)* $(1/3, 5)$
(c) $(5, \infty)$ *(d)* $(-\infty, 1/3) \cup (5, \infty)$

9. The solution set of the inequation $||x|-1|<|1-x|$, $x \in R$ is
(a) $(-1, 1)$ *(b)* $(0, \infty)$ *(c)* $(-1, \infty)$ *(d)* $(-\infty, 0)$

10. The set of all real numbers x for which $x^2-|x+2|+x>0$, is
(a) $(-\infty, -2) \cup (2, \infty)$ *(b)* $(-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$
(c) $(-\infty, -1) \cup (1, \infty)$ *(d)* $(\sqrt{2}, \infty)$

11. The area of the region represented by $|x-y| \le 3$ and $|x+y| \le 3$, is
(a) 36 sq units *(b)* 18 sq units
(c) 72 sq units *(d)* 6 sq units

12. The solution set of the inequation $|[|x|-7]|-5<0$, is ([.] denotes the greatest integer function)
(a) $[3, 12)$
(b) $(-12, -3]$
(c) $(-12, 12)$
(d) $(-12, -3] \cup [3, 12)$

13. The set of real values of x satisfying the inequality $|x^2+x-6|<6$, is
(a) $(-4, 3)$
(b) $(-3, 2)$
(c) $(-4, -3) \cup (2, 3)$
(d) $(-4, 1) \cup (0, 3)$

14. If $x^2-ax+1-2a^2>0$, $\forall\, x \in R$, then a belongs to
(a) $(-2/3, 2/3)$ *(b)* $[-2/3, 2/3]$
(c) $(-2/3, 1)$ *(d)* $(0, 2/3)$

15. If $9^{x+1}+(a^2-4a-2)3^x+1>0$, $\forall\, x \in R$, then
(a) $a \in R-\{2\}$ *(b)* $a \in R^+$
(c) $a \in [1, \infty)$ *(d)* $a \in R$

16. If $x(x+2)^2(x-1)^5(2x-3)(x-3)^4 \ge 0$, then x belongs to
(a) $[0, 1] \cup [3/2, \infty)$ *(b)* $(0, 1) \cup (3/2, \infty)$
(c) $(-1, 0) \cup (3/2, \infty)$ *(d)* $[-1, 0] \cup [3/2, \infty)$

17. The solution set of the inequality $x+\sqrt{x} \ge \sqrt{x}-3$ is
(a) $[-3, \infty)$ *(b)* $[0, \infty)$ *(c)* $(-\infty, \infty)$ *(d)* $[3, \infty)$

18. The value of x which satisfies $\sqrt{x+3-4\sqrt{x-1}}+\sqrt{x+8-6\sqrt{x-1}}=1$ is

(a) $[5, \infty)$ (b) $(-\infty, 10]$
(c) $[5, 10]$ (d) $[-10, -5]$

19. If $\left(\frac{1}{2}\right)^{\frac{|x+2|}{2-|x|}} > 9$, then x belongs to

(a) $(2, 6)$ (b) $[2, 6]$
(c) $(-6, -2)$ (d) $[-6, -2]$

20. If $\left|\frac{x+1}{x}\right|+|x+1|=\frac{(x+1)^2}{|x|}$, then x belongs to

(a) $(0, \infty)$ (b) $(-\infty, 0)$
(c) $(0, \infty) \cup \{-1\}$ (d) $(-\infty, \infty)$

Answers

1. (a)	2. (d)	3. (b)	4. (c)	5. (c)
6. (c)	7. (d)	8. (b)	9. (d)	10. (b)
11. (b)	12. (d)	13. (d)	14. (a)	15. (a)
16. (a)	17. (b)	18. (c)	19. (a)	20. (c)

LOGARITHMS

If a is a positive real number, other than 1 and x is a rational number such that $a^x = N$, then we say that logarithm of N to base a is x or x is the logarithm of N to the base a, written as $\log_a N = x$. Thus, $a^x = N \Leftrightarrow \log_a N = x$

Laws of Logarithms

Following are the fundamental laws of logarithms to any base a, where $a > 0$ and $a \neq 1$.

(i) If m, n are positive rational numbers, then
$\log_a(mn) = \log_a m + \log_a n$
In general,
$\log_a(x_1\ x_2 \ldots x_n) = \log_a x_1 + \log_a x_2 + \ldots + \log_a x_n$

(ii) If m, n are positive rational numbers, then $\log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n$

(iii) If m, n are positive rational numbers, then $\log_a(m^n) = n \log_a m$

(iv) $\log_a 1 = 0$

(v) If m is positive rational number and a, b are positive real numbers such that $a \neq 1, b \neq 1$, then
$\log_a m = \frac{\log_b m}{\log_b a}$ or $\log_a m \times \log_b a = \log_b m$

REMARK

Replacing b by m in above, we get
$\log_a m = \frac{\log_m m}{\log_m a} \Rightarrow \log_a m = \frac{1}{\log_m a} \Rightarrow \log_a m \times \log_m a = 1$

(vi) $\log_a a = 1$

(vii) If a is a positive real number and n is a positive rational number, then $a^{\log_a n} = n$

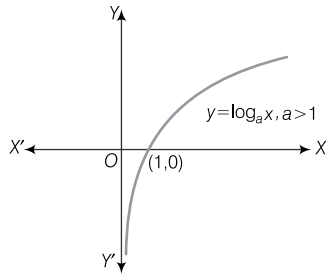
(viii) If a is a positive real number and n is a positive rational number, then $\log_{a^q} n^p = \frac{p}{q}\log_a n$

(ix) If x, y and $a(\neq 1)$ are positive real numbers, then
$x^{\log_a y} = y^{\log_a x}$

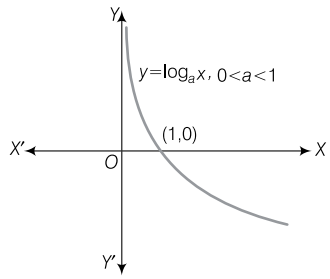
Some Useful Results

Following are some useful results on logarithms

1. If $a > 1$, then
   (i) $\log_a x < 0, \forall x \in (0, 1)$ (ii) $\log_a x = 0$ for $x = 1$
   (iii) $\log_a x > 0$ for $x > 1$
   (iv) $x > y \Leftrightarrow \log_a x > \log_a y$, i.e. $\log_a x$ is an increasing function.
   The graph of $y = \log_a x$, for $a > 1$ is as shown below



2. If $0 < a < 1$, then
   (i) $\log_a x < 0, \forall x > 1$ (ii) $\log_a x = 0$ for $x = 1$
   (iii) $\log_a x > 0, \forall x \in (0, 1)$
   (iv) $x > y \Leftrightarrow \log_a x < \log_a y$, i.e. $\log_a x$ is a decreasing function.
   The graph of $y = \log_a x$, $0 < a < 1$ is as shown below



3. If $0 < a < 1$, then
   (i) $\log_a x > 1$, if $0 < x < a$ (ii) $\log_a x = 1$, if $x = a$
   (iii) $\log_a x < 1$, if $x > a$
4. If $a > 1$, then
   (i) $\log_a x > 1$, if $x > a$ (ii) $\log_a x < 1$, if $0 < x < a$
   (iii) $\log_a x = 1$, if $x = a$

MASTER STROKES

1. If $0 \le a \le x$, then the minimum value of $\log_a x + \log_x x$ is
(a) 1 (b) 2
(c) 0 (d) None of these

2. If $x = \log_{2a} a$, $y = \log_{3a} 2a$ and $z = \log_{4a} 3a$, then $xyz + 1$ is equal to
(a) $2yz$ (b) $2xy$
(c) $2zx$ (d) None of these

3. If a, b, c are positive real numbers, then $\frac{1}{\log_a bc + 1} + \frac{1}{\log_b ca + 1} + \frac{1}{\log_c ab + 1}$ is equal to
(a) 0 (b) 1 (c) 2 (d) -1

4. If $4^x + 2^{2x-1} = 3^{x+\frac{1}{2}} + 3^{x-\frac{1}{2}}$, then x is equal to
(a) $\frac{1}{2}$ (b) $\frac{3}{2}$ (c) $\frac{5}{2}$ (d) 1

5. If $\frac{\log x}{2a + 3b - 5c} = \frac{\log y}{2b + 3c - 5a} = \frac{\log z}{2c + 3a - 5b}$, then xyz is equal to
(a) 2 (b) 1 (c) 0 (d) -1

6. $\frac{1}{\log_2 n} + \frac{1}{\log_3 n} + \frac{1}{\log_4 n} + \ldots + \frac{1}{\log_{43} n}$ is equal to
(a) 1 (b) $\log_{43!} n$
(c) $\log_n 43!$ (d) None of these

7. If a, b, c are the sides of a right angled triangle in which $c > a$, b and $c - b \ne 1$, $c + b \ne 1$, then the value of $\frac{\log_{c+b} a + \log_{c-b} a}{\log_{c+b} a \cdot \log_{c-b} a}$ is
(a) 1 (b) 2
(c) $\frac{1}{2}$ (d) None of these

8. If $\log_{ax} x$, $\log_{bx} x$, $\log_{cx} x$ are in HP, where a, b, c, x belong to $(1, \infty)$, then a, b, c are in
(a) AP (b) GP
(c) HP (d) None of these

9. The solution set of $\log_2 |4 - 5x| > 2$, is
(a) $(8/5, \infty)$ (b) $(4/5, 8/5)$
(c) $(-\infty, 0) \cup (8/5, \infty)$ (d) None of these

10. If $\log_{0.3}(x - 1) < \log_{0.09}(x - 1)$, then x lies in the interval
(a) $(2, \infty)$ (b) $(-2, -1)$
(c) $(1, 2)$ (d) None of these

11. The solution set of the equation $\log_x 2 \cdot \log_{2x} 2 = \log_{4x} 2$, is
(a) $\{2^{-\sqrt{2}}, 2^{\sqrt{2}}\}$ (b) $\left\{\frac{1}{2}, 2\right\}$
(c) $\left\{\frac{1}{4}, 4\right\}$ (d) None of these

12. The set of values of x for which $\log_e x > \frac{x-2}{x}$, is
(a) $(1, \infty)$ (b) $(1, 2)$
(c) R (d) $(2, \infty)$

13. If $[.]$ denotes the greatest integer function, then the value of natural number n satisfying the equation $[\log_2 1] + [\log_2 2] + [\log_2 3] + \ldots + [\log_2 n] = 1538$, is
(a) 255 (b) 256 (c) 254 (d) 313

14. If $\log_{\cos x} \sin x \ge 2$ and $x \in [0, 3\pi]$, then $\sin x$ lies in the interval
(a) $\left[\frac{\sqrt{5}-1}{2}, 1\right]$ (b) $\left(0, \frac{\sqrt{5}-1}{2}\right)$
(c) $[0, 1/2]$ (d) None of these

15. If $n = 1999!$, then $\sum_{x=1}^{1999} \log_n x$ is equal to
(a) 1 (b) 0
(c) $\sqrt[1999]{1999}$ (d) -1

16. If $\log_{10}(x^3 + y^3) - \log_{10}(x^2 - xy + y^2) \le 2$, then the maximum value of xy, $\forall\ x > 0$, $y > 0$ is
(a) 2500 (b) 3000 (c) 1200 (d) 3500

17. The value of $(0.16)^{\log_{2.5}\left(\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \ldots \infty\right)}$ is equal to
(a) 1 (b) 2
(c) 3 (d) 4

18. If $9^{\log_3(\log_2 x)} = \log_2 x - (\log_2 x)^2 + 1$, then x is equal to
(a) -1 (b) 0
(c) 1 (d) 2

19. Let $x \in (1, \infty)$ and n be a positive integer greater than 1. If $f_n(x) = \frac{n}{\frac{1}{\log_2 x} + \frac{1}{\log_3 x} + \ldots + \frac{1}{\log_n x}}$, then $(n!)^{f_n(x)}$ is equal to
(a) n^x (b) x^n
(c) n^n (d) n^{nx}

20. If $\log_2 \sin x - \log_2 \cos x - \log_2(1 - \tan^2 x) = -1$, then x is equal to
(a) $\frac{n\pi}{2} + \frac{\pi}{8}, n \in Z$ (b) $n\pi - \frac{\pi}{8}, n \in Z$
(c) $\frac{n\pi}{4} + \frac{\pi}{2}, n \in Z$ (d) None of these

Answers

1. (b)	**2.** (a)	**3.** (b)	**4.** (b)	**5.** (b)
6. (c)	**7.** (b)	**8.** (b)	**9.** (c)	**10.** (a)
11. (a)	**12.** (d)	**13.** (a)	**14.** (b)	**15.** (a)
16. (a)	**17.** (d)	**18.** (d)	**19.** (b)	**20.** (a)

QUADRATIC EQUATION

Let $a_0, a_1, a_2, \ldots, a_n$ be real numbers and x be a real variable. Then, $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ is called a real polynomial of real variable x with real coefficients.

Descarte's Rule of Signs

The maximum number of positive real roots of a polynomial equation $f(x) = 0$ is the number of changes of signs from positive to negative and negative to positive in $f(x)$.

Relation between Roots and Coefficients

If $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ are roots of the equation $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n = 0$

Then, $f(x) = a_0(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\ldots(x-\alpha_n)$

$\therefore\ a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n$
$= a_0(x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_n)$

On comparing the coefficients of like powers of x on both sides, we get

$$S_1 = \alpha_1 + \alpha_2 + \ldots + \alpha_n = \Sigma\alpha_i = \frac{-a_1}{a_0}$$

or $$S_1 = -\frac{\text{coefficient of } x^{n-1}}{\text{coefficient of } x^n}$$

$$S_2 = \alpha_1\alpha_2 + \alpha_1\alpha_3 + \ldots = \sum_{i\neq j}\alpha_i\alpha_j = (-1)^2\frac{a_2}{a_0}$$

or $$S_2 = \frac{(-1)^2 \text{ coefficient of } x^{n-2}}{\text{coefficient of } x^n}$$

$$S_3 = \alpha_1\alpha_2\alpha_3 + \alpha_2\alpha_3\alpha_4 + \ldots = \sum_{i\neq j\neq k}\alpha_i\alpha_j\alpha_k = (-1)^3\frac{a_3}{a_0}$$

or $$S_3 = \frac{(-1)^3\text{coefficient of } x^{n-3}}{\text{coefficient of } x^n}$$

and so on.

$$\therefore\ S_n = \alpha_1\alpha_2\alpha_3\ldots\alpha_n = (-1)^n\frac{a_n}{a_0} = \frac{(-1)^n \text{ constant term}}{\text{coefficient of } x^n}$$

Here, S_k denotes the sum of the product of the roots taken k at a time.

Particular Cases

1. Quadratic Equation

If α, β are the roots of the quadratic equation $ax^2 + bx + c = 0$, then $\alpha + \beta = \frac{-b}{a}$ and $\alpha\beta = \frac{c}{a}$.

2. Cubic Equation

If α, β and γ are the roots of the cubic equation $ax^3 + bx^2 + cx + d = 0$, then

$$\alpha + \beta + \gamma = \frac{-b}{a}$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = (-1)^2\frac{c}{a} = \frac{c}{a}$$

and $$\alpha\beta\gamma = (-1)^3\frac{d}{a} = \frac{-d}{a}$$

3. Biquadratic Equation

If α, β, γ and δ are the roots of the biquadratic equation $ax^4 + bx^3 + cx^2 + dx + e = 0$, then

$$\alpha + \beta + \gamma + \delta = \frac{-b}{a}$$

$$\alpha\beta + \beta\gamma + \gamma\delta + \delta\alpha + \beta\delta + \alpha\gamma = (-1)^2\frac{c}{a} = \frac{c}{a}$$

$$\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \alpha\beta\delta = (-1)^3\frac{d}{a} = \frac{-d}{a}$$

and $$\alpha\beta\gamma\delta = (-1)^4\frac{e}{a} = \frac{e}{a}$$

Formation of a Polynomial Equation

If $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ are the roots of an nth degree equation, then the equation is given by

$$x^n - S_1x^{n-1} + S_2x^{n-2} - S_3x^{n-3} + \ldots + (-1)^n S_n = 0$$

Where, S_k denotes the sum of the products of roots taken k at a time.

Particular Cases

1. Quadratic Equation

If α, β are the roots of the quadratic equation, then the equation is

$$x^2 - S_1x + S_2 = 0$$

i.e. $$x^2 - (\alpha + \beta)x + \alpha\beta = 0$$

2. Cubic Equation

If α, β and γ are the roots of a cubic equation, then the equation is

$$x^3 - S_1x^2 + S_2x - S_3 = 0$$

i.e. $x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \beta\gamma + \gamma\alpha)x - \alpha\beta\gamma = 0$

3. Biquadratic Equation

If α, β, γ and δ are the roots of a biquadratic equation, then the equation is

$$x^4 - S_1x^3 + S_2x^2 - S_3x + S_4 = 0$$

i.e. $x^4 - (\alpha + \beta + \gamma + \delta)x^3$
$+ (\alpha\beta + \beta\gamma + \gamma\delta + \alpha\delta + \beta\delta + \alpha\gamma)x^2$
$- (\alpha\beta\gamma + \alpha\beta\delta + \beta\gamma\delta + \gamma\delta\alpha)x + \alpha\beta\gamma\delta = 0$

Best Practice SHOTS

1. The harmonic mean of the roots of the equation $(5+\sqrt{2})x^2-(4+\sqrt{5})x+(8+2\sqrt{5})=0$ is

(a) 2 (b) 4 (c) 7 (d) 8

2. If the equation $x^3-px^2+qx-r=0$ has roots in AP, then

(a) $2p^3=9pq-27r$ (b) $2p^3=9pq+27r$
(c) $2q^3=9pq-27r$ (d) $p^3=9pq-27r$

3. If x^2+x+1 is a factor of ax^3+bx^2+cx+d, then the real root of $ax^3+bx^2+cx+d=0$ is

(a) $\frac{d}{a}$ (b) $\frac{-d}{a}$ (c) $\frac{-b}{a}$ (d) $\frac{-c}{a}$

4. If A, G and H are respectively the AM, GM and HM of three positive numbers a, b and c, then the equation whose roots are a, b, c is

(a) $x^3-3Ax^2+\frac{3G^3}{H}x-G^3=0$
(b) $x^3+3Ax^2+\frac{3G^3}{H}x-G^3=0$
(c) $x^3+Ax^2+\frac{G^3x}{H}-G^3=0$
(d) $x^3-3Ax^2+\frac{3G^3}{H}x-G^3=0$

5. If α,β are roots of the equation $2x^3+6x+b=0$ $(b<0)$, then $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}$ is less than

(a) 2 (b) -4
(c) -2 (d) 4

Answers

1. (b) **2.** (a) **3.** (b) **4.** (a) **5.** (c)

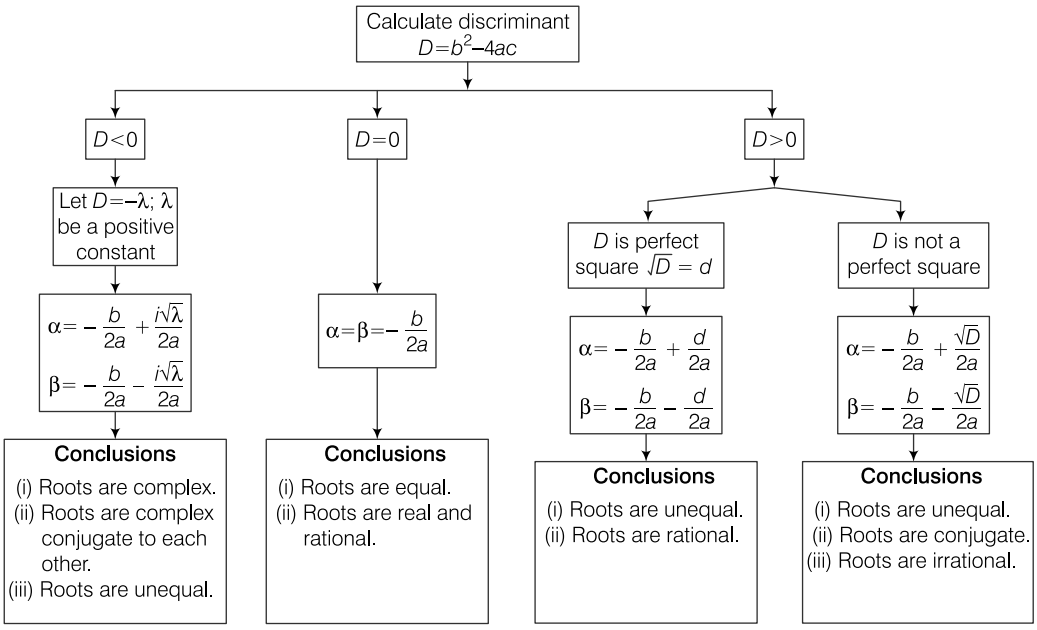
NATURE OF ROOTS OF QUADRATIC EQUATION

Quadratic Equation

An equation of the form $ax^2+bx+c=0$ …(i)

where $a\neq 0$, $a,b,c\in R$ is called a quadratic equation with real coefficients. The quantity $D=b^2-4ac$ is known as the **discriminant** of the quadratic equation whose roots are given by $\alpha=\frac{-b+\sqrt{b^2-4ac}}{2a}$ and $\beta=\frac{-b-\sqrt{b^2-4ac}}{2a}$

The nature of roots of a quadratic equation can be determine with the help of following diagram



Some Important Results

If roots of a quadratic equation $ax^2 + bx + c = 0, a \neq 0$ are α and β, then

(i) $\alpha - \beta = \sqrt{(\alpha+\beta)^2 - 4\alpha\beta} = \pm\dfrac{\sqrt{b^2-4ac}}{a} = \pm\dfrac{\sqrt{D}}{a}$

(ii) $\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta = \dfrac{b^2-2ac}{a^2}$

(iii) $\alpha^2 - \beta^2 = (\alpha+\beta)\sqrt{(\alpha+\beta)^2-4\alpha\beta}$

$= \pm\dfrac{b\sqrt{b^2-4ac}}{a^2} = \dfrac{\pm b\sqrt{D}}{a^2}$

(iv) $\alpha^3 + \beta^3 = (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta) = \dfrac{-b(b^2-3ac)}{a^3}$

(v) $\dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha} = \dfrac{\alpha^2+\beta^2}{\alpha\beta} = \dfrac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta} = \dfrac{b^2-2ac}{ac}$

(vi) $\alpha^3 - \beta^3 = (\alpha-\beta)^3 + 3\alpha\beta(\alpha-\beta)$

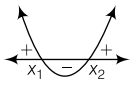
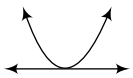
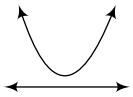
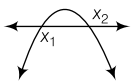
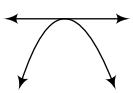
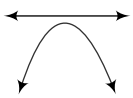
$= \pm\dfrac{(b^2-ac)\sqrt{b^2-4ac}}{a^3}$

(vii) $\alpha^2\beta + \beta^2\alpha = \alpha\beta(\alpha+\beta) = \dfrac{-bc}{a^2}$

Sign of Quadratic Expression and their Graphs

Let $f(x) = ax^2 + bx + c$ and $D = b^2 - 4ac$. The shape of the curve $y = f(x)$, is a parabola having vertex at $\left(\dfrac{-b}{2a}, \dfrac{-D}{4a}\right)$.

Following are six possible cases, arises

***Case* I** When $a > 0, D > 0$		(i) $y > 0$, when $x < x_1$ and $x > x_2$, i.e. graph of the curve is above X-axis (ii) $y < 0$, when $x \in (x_1, x_2)$ (iii) $y = 0$, when $x = x_1, x_2$
***Case* II** When $a > 0, D = 0$		(i) $y \geq 0, \forall\, x \in R$ (ii) $y = 0$, at $x = \dfrac{-b}{2a}$
***Case* III** When $a > 0, D < 0$		(i) $y > 0, \forall\, x \in R$ (ii) Roots are imaginary.
***Case* IV** When $a < 0, D > 0$		(i) $y > 0$, when $x \in (x_1, x_2)$ (ii) $y = 0$, when $x = x_1, x_2$ (iii) $y < 0$, when $x < x_1$ or $x > x_2$
***Case* V** When $a < 0, D = 0$		(i) $y \leq 0, \forall\, x \in R$ (ii) $y = 0$ at $x = \dfrac{-b}{2a}$
***Case* VI** When $a < 0, D < 0$		(i) $y < 0, \forall x \in R$ (ii) Roots are imaginary.

Transformation of Quadratic Equation

If $ax^2 + bx + c = 0$ is a quadratic equation, then following rules are very helpful to transform the quadratic equation.

(i) To form an equation whose roots are $\lambda(\neq 0)$ times the roots of $ax^2 + bx + c = 0$, replace x by x/λ.

(ii) To form an equation whose roots are the negative of the roots of $ax^2 + bx + c = 0$, replace x by $-x$.

(iii) To form an equation whose roots are reciprocal to given equation replace x by $\dfrac{1}{x}$.

(iv) To form an equation whose roots are λ more than the roots of the given equation replace x by $(x - \lambda)$.

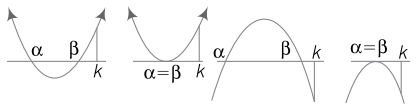
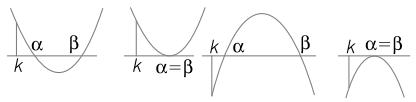
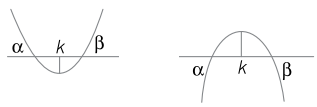
(v) An equation whose roots are squares of the roots of a given equation is obtained by replacing x by $\sqrt{x}$ in the given equation.

(vi) An equation whose roots are cubes of the roots of a given equation is obtained by replacing x by $x^{1/3}$ in the given equation.

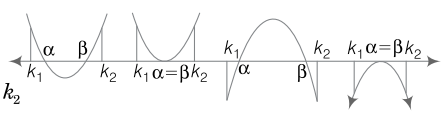
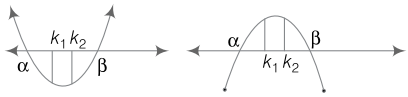
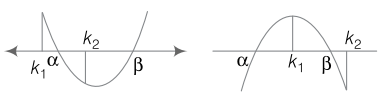
Position of Roots of a Quadratic Equation

Let $ax^2 + bx + c = 0$ be a quadratic equation whose roots are α and β, then

1. Roots with respect to one real number k

Situation	Required condition	Graphical situation
$\alpha < \beta < k$ [roots are less than k]	$D \geq 0$, $af(k) > 0$, $k > \dfrac{-b}{2a}$	
$k < \alpha < \beta$ [k is less than roots]	$D \geq 0$, $af(k) > 0$, $k < \dfrac{-b}{2a}$	
$\alpha < k < \beta$ [k lies between the roots]	$D > 0$, $af(k) < 0$	

2. Roots with respect to two real numbers k_1 and k_2

Situation	Required condition	Graphical situation
$k_1 < \alpha < \beta < k_2$ [roots are between k_1 and k_2]	$D \geq 0$, $af(k_1) > 0$, $af(k_2) > 0$, $k_1 < \dfrac{-b}{2a} < k_2$	
$\alpha < k_1 < k_2 < \beta$ [k_1 and k_2 lie between the roots]	$D > 0$, $af(k_1) < 0$, $af(k_2) < 0$	
$k_1 < \alpha < k_2 < \beta$ [one root lies between k_1 and k_2]	$D > 0$, $f(k_1)\, f(k_2) < 0$	

Common Roots

Let $a_1x^2 + b_1x + c_1 = 0$ and $a_2x^2 + b_2x + c_2 = 0$ be two quadratic equations such that $a_1, a_2 \neq 0$ and $a_1b_2 \neq a_2b_1$. Let α be the common root of these two equations. Then,

$a_1\alpha^2 + b_1\alpha + c_1 = 0$ and $a_2\alpha^2 + b_2\alpha + c_2 = 0$

On solving, we get $\frac{\alpha^2}{b_1c_2 - b_2c_1} = \frac{\alpha}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - a_2b_1}$

$\Rightarrow \quad \alpha^2 = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}$ and $\alpha = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}$

Hence, condition for one common root is

$$(b_1c_2 - b_2c_1)(a_1b_2 - a_2b_1) = (c_1a_2 - c_2a_1)^2$$

If the two equations have both roots common, then

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

Best Practice SHOTS

6. The real values of a for which the quadratic equation $2x^2 - (a^3 + 8a - 1)\,x + a^2 - 4a = 0$ possesses roots of opposite signs are given by

(a) $a > 5$ *(b)* $0 < a < 4$ *(c)* $a > 0$ *(d)* $a > 7$

7. If α, β are the roots of $ax^2 + bx + c = 0$; $\alpha + h, \beta + h$ are the roots of $px^2 + qx + r = 0$ and D_1, D_2 the respective discriminants of these equations, then $D_1 : D_2$ is equal to

(a) $\frac{a^2}{p^2}$ *(b)* $\frac{b^2}{q^2}$

(c) $\frac{c^2}{r^2}$ *(d)* None of these

8. If every pair from among the equations $x^2 + px + qr = 0$, $x^2 + qx + rp = 0$ and $x^2 + rx + pq = 0$ has a common root, then the sum of the three common root is

(a) $2(p + q + r)$ *(b)* $-(p + q + r)$

(c) $p + q + r$ *(d)* pqr

9. If α and β are the roots of the equation $x^2 + ax + b = 0$ and α^4 and β^4 are the roots of $x^2 - px + q = 0$, then roots of $x^2 - 4bx + 2b^2 - p = 0$ are always

(a) both non-real *(b)* both positive

(c) both negative *(d)* positive and negative

10. If $b > a$, then the equation $(x - a)(x - b) - 1 = 0$, has

(a) both roots in $[a, b]$

(b) both roots in $(-\infty, a]$

(c) both roots in (b, ∞)

(d) one root in $(-\infty, a)$ and other in (b, ∞)

Answers

6. (b) **7.** (a) **8.** (c) **9.** (d) **10.** (d)

MASTER STROKES

1. If $a \in R$ and $a_1, a_2, a_3, \ldots, a_n \in R$, then $(x - a_1)^2 + (x - a_2)^2 + \ldots + (x - a_n)^2$ assumes its least value at x which is equal to

(a) $a_1 + a_2 + \ldots + a_n$ *(b)* $2(a_1 + a_2 + \ldots + a_n)$

(c) $n(a_1 + a_2 + \ldots + a_n)$ *(d)* $\frac{a_1 + a_2 + \ldots + a_n}{n}$

2. The number of real roots of $(6 - x)^4 + (8 - x)^4 = 16$ is

(a) 0 *(b)* 2 *(c)* 4 *(d)* 6

3. If α, β, γ and σ are the roots of the equation $x^4 + 4x^3 - 6x^2 + 7x - 9 = 0$, then the value of $(1 + \alpha^2)(1 + \beta^2)(1 + \gamma^2)(1 + \sigma^2)$ is

(a) 5 *(b)* 9 *(c)* 11 *(d)* 13

4. Let Δ^2 be the discriminant and α, β be the roots of the equation $ax^2 + bx + c = 0$. Then, $2a\alpha + \Delta$ and $2a\beta - \Delta$ can be the roots of the equation

(a) $x^2 + 2bx + b^2 = 0$

(b) $x^2 - 2bx + b^2 = 0$

(c) $x^2 + 2bx - 3b^2 - 16ac = 0$

(d) $x^2 - 2bx - 3b^2 + 16ac = 0$

5. If the ratio of the roots of the equation $ax^2 + bx + c = 0$ is equal to the ratio of the roots of the equation $x^2 + x + 1 = 0$, then a, b, c are in

(a) AP

(b) GP

(c) HP

(d) None of the above

6. If ${}^6C_k + 2 \cdot {}^6C_{k+1} + {}^6C_{k+2} > {}^8C_3$, then the quadratic equation whose roots are α, β and $\alpha^{k-1}, \beta^{k-1}$ have

(a) no common root *(b)* one common root

(c) both common roots *(d)* imaginary roots

7. If both roots of the equation $x^2 - 2ax + a^2 - 1 = 0$ lie between $(-2, 2)$, then $[a]$ can be

(a) $-1, 0$ *(b)* $0, 1$

(c) $1, 2$ *(d)* None of these

8. If the equation $ax^2 + bx + 6 = 0$ has real roots, where $a \in R$, $b \in R$, then the greatest value of $3a + b$ is

(a) 4 *(b)* -1

(c) -2 *(d)* 1

9. If $a \cdot 3^{\tan x} + a \cdot 3^{-\tan x} - 2 = 0$ has real solutions, $x \neq \pi/2$, $0 \leq x \leq \pi$, then the set of possible values of the parameter 'a' is
(a) $[-1, 1]$ *(b)* $[-1, 0]$
(c) $(0, 1]$ *(d)* $(0, \infty)$

10. Set of values of a for which both roots of the equation $x^2 - 2x - a^2 = 0$ lie between the roots of the equation $x^2 - 2x + a^2 - 11a + 12 = 0$, is
(a) $(1, 4)$ *(b)* $(3/2, 4)$
(c) $(-4, 4)$ *(d)* $(1, 2)$

11. Let $f(x) = ax^2 + bx + c$; $a, b, c \in R$; $a \neq 0$ satisfying $f(1) + f(2) = 0$. Then, the quadratic equation $f(x) = 0$ has
(a) no real root
(b) 1 and 2 as real roots
(c) two equal roots
(d) two distinct real roots

12. If $f(x) = x^2 + 2bx + 2c^2$ and $g(x) = -x^2 - 2cx + b^2$ be such that min $f(x) >$ max $g(x)$, then the relation between b and c, is
(a) no real values of b and c *(b)* $0 < c < \sqrt{2}\, b$
(c) $|c| < \sqrt{2}\,|b|$ *(d)* $|c| > \sqrt{2}\,|b|$

13. If α, β and γ are the roots of the equation $x^3 + 4x + 1 = 0$, then $(\alpha + \beta)^{-1} + (\beta + \gamma)^{-1} + (\gamma + \alpha)^{-1}$ is equal to
(a) 2 *(b)* 3 *(c)* 4 *(d)* 5

14. The smallest value of k for which both the roots of the equation $x^3 - 8kx + 16(k^2 - k + 1) = 0$ are real, distinct and have values atleast 4, is
(a) 2 *(b)* 3
(c) 4 *(d)* 5

15. Let α, β be the roots of the equation $x^2 - px + r = 0$ and $\frac{\alpha}{2}, 2\beta$ be the roots of the equation $x^2 - qx + r = 0$. Then, the value of r is
(a) $\frac{2}{9}(p - q)(2q - p)$ *(b)* $\frac{2}{9}(q - p)(2p - q)$
(c) $\frac{2}{9}(q - 2p)(2q - p)$ *(d)* $\frac{2}{9}(2p - q)(2q - p)$

16. Let $f(x) = ax^2 + bx + c$, $a \neq 0$ and $\Delta = b^2 - 4ac$. If $\alpha + \beta, \alpha^2 + \beta^2$ and $\alpha^3 + \beta^3$ are in GP, then
(a) $\Delta \neq 0$ *(b)* $b\Delta = 0$
(c) $c\Delta = 0$ *(d)* $bc \neq 0$

17. Let a, b and c be three real numbers satisfying $[a\ b\ c]\begin{bmatrix} 1 & 9 & 7 \\ 8 & 2 & 7 \\ 7 & 3 & 7 \end{bmatrix} = [0\ 0\ 0]$ and α, β be the roots of the equation $ax^2 + bx + c = 0$, then $\sum_{n=0}^{\infty}\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)^n$ is
(a) 6 *(b)* 7 *(c)* $\frac{6}{7}$ *(d)* ∞

18. Let α and β be the roots of equation $x^2 - 6x - 2 = 0$. If $a_n = \alpha^n - \beta^n$ for $n \geq 1$, then the value of $\frac{a_{10} - 2a_8}{2a_9}$ is equal to
(a) 3 *(b)* -3
(c) 6 *(d)* -6

19. If the roots of the equation $\frac{1}{x + p} + \frac{1}{x + q} = \frac{1}{r}$ are equal in magnitude and opposite in sign, then the product of roots is
(a) $\frac{-1}{2}(p^2 - q^2)$ *(b)* $p^2 + q^2$
(c) $\frac{1}{2}(p^2 - q^2)$ *(d)* $\frac{-1}{2}(p^2 + q^2)$

20. Let α and β be the roots of the equation $px^2 + qx + r = 0$, $p \neq 0$. If p, q, r are in AP and $\frac{1}{\alpha} + \frac{1}{\beta} = 4$, then the value of $|\alpha - \beta|$ is
(a) $\frac{\sqrt{34}}{9}$ *(b)* $\frac{2\sqrt{13}}{9}$
(c) $\frac{\sqrt{61}}{9}$ *(d)* $\frac{2\sqrt{17}}{9}$

21. If the roots of the quadratic equation $x^2 + px + q = 0$ are $\tan 30°$ and $\tan 15°$, respectively, then the value of $2 + q - p$ is equal to
(a) 1 *(b)* 2
(c) 3 *(d)* 4

22. The values of the parameter a such that the roots of the equation $2x^2 + 6x + a = 0$ satisfy the inequality $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} < 2$ is
(a) $a < 0$ *(b)* $a > \frac{9}{2}$
(c) Both (a) and (b) *(d)* None of these

23. If α and β are the roots of the equations $x^2 - ax + b = 0$ and $A_n = \alpha^n + \beta^n$, then which of the following is true?
(a) $A_{n+1} = aA_n + bA_{n-1}$ *(b)* $A_{n+1} = bA_n + aA_{n-1}$
(c) $A_{n+1} = aA_n - bA_{n-1}$ *(d)* $A_{n+1} = bA_n - aA_{n-1}$

24. If α and β be the roots of the equation $x^2 + px - \frac{1}{2p^2} = 0$, where $p \in R$. Then, the minimum value of $\alpha^4 + \beta^4$ is
(a) $2\sqrt{2}$ *(b)* $2 - \sqrt{2}$
(c) 2 *(d)* $2 + \sqrt{2}$

Answers

1. (d)	**2.** (b)	**3.** (d)	**4.** (a)	**5.** (b)
6. (c)	**7.** (a)	**8.** (c)	**9.** (c)	**10.** (b)
11. (d)	**12.** (d)	**13.** (c)	**14.** (a)	**15.** (d)
16. (c)	**17.** (b)	**18.** (a)	**19.** (d)	**20.** (b)
21. (c)	**22.** (c)	**23.** (c)	**24.** (d)	

SETS

A set is a well defined collection of objects. Here, term 'well defined' means that there is some definite rule on the basis of which one can decide whether an object is in the collection or not.

Representation of Set

A set can be represent in following two ways.

1. **Roster form** In this method, a set is described by listing elements, separated by commas, within braces {}.
2. **Set builder form** In this method, a set is described by a characterising property, say $P(x)$, of its elements x. In such a case, the set is described by $\{x : P(x)$ holds$\}$.

Types of Sets

Following are the types of sets

1. **Empty set** A set is said to be empty or null or void set, if it has no element and it is denoted by ϕ or {}.
2. **Singleton set** A set consisting of a single element is called a singleton set.
3. **Finite set** A set is called a finite set, if it is either a void set or its element can be listed by natural numbers 1, 2, 3, ... and the process of listing terminates at a certain natural number n(say).

Note The number n in the above definition is called the cardinal number or order of a finite set A and is denoted by $n(A)$.

4. **Subset** Let A and B be two sets. If every element of set A is an element of B, then A is called a subset of B and we write $A \subseteq B$. If A is a subset of B, we also say that B contains A or B is super set of A and we write $B \supseteq A$.
   Clearly, empty set is a subset of every set. Also, every set is a subset of itself.
   If a set has n elements, then the total number of its subsets is 2^n.
   A subset A of a set B is called a proper subset of B, if $A \neq B$ and we write $A \subset B$.
5. **Power set** The set of all subsets of a set A is called the power set of A and is denoted by $P(A)$.

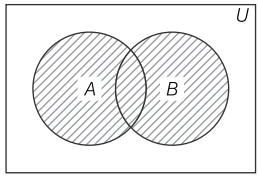
Note If $A = \phi$, then $P(A) = \{\phi\}$.

6. **Equal sets** Two finite sets are equal, if every element of A is a member of B and every element of B is a member of A, i.e. $A \subseteq B$ and $B \subseteq A$.
7. **Universal set** A set that contains all sets in a given context is called the universal set.

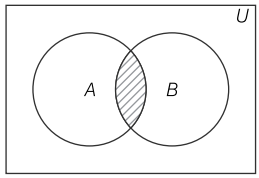
Operations on Sets

Following are operations on sets

1. **Union of sets** The union of two sets A and B is the set of all those elements which belong to either A or B or both A and B.
   It is denoted by $A \cup B$.
   Thus, $A \cup B = \{x : x \in A$ or $x \in B\}$



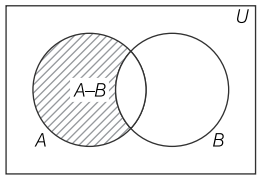
2. **Intersection of sets** The intersection of two sets A and B is the set of all those elements that belong to both A and B.
   It is denoted by $A \cap B$.



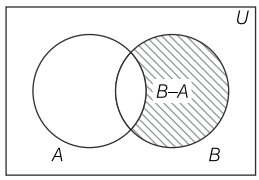
   Thus, $A \cap B = \{x : x \in A$ and $x \in B\}$

Note **Disjoint sets** Two sets A and B are said to be disjoint, if $A \cap B = \phi$, i.e. they do not have any common element.

3. **Difference of sets** Let A and B be two sets. The difference of A and B, written as $A - B$, is the set of all those elements of A which do not belong to B.

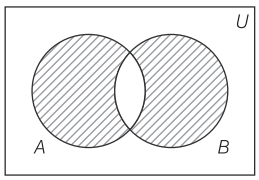


   Thus, $A - B = \{x : x \in A$ and $x \notin B\}$
   The difference $B - A$ is the set of all those elements of B that do not belong to A, i.e. $B - A = \{x : x \in B$ and $x \notin A\}$.

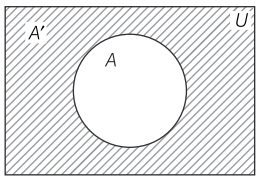


4. **Symmetric difference** Let A and B be two sets. The symmetric difference of sets A and B is the set $(A - B) \cup (B - A)$ and is denoted by $A \Delta B$.

Thus, $A\Delta B = (A - B) \cup (B - A)$
$= \{x : x \notin A \cap B\}$
$= (A \cup B) - (A \cap B)$



5. **Complement of a set** Let U be the universal set and let A be a set such that $A \subset U$. Then, the complement of A with respect to U is denoted by A' or A^C or $U - A$ and is defined, the set of all those elements of U which are not in A.



Thus, $A' = \{x \in U : x \notin A\}$. If $x \in A' \Leftrightarrow x \notin A$
Also,

(i) $U' = \phi$ (ii) $\phi' = U$
(iii) $(A')' = A$ (iv) $A \cup A' = U$
(v) $A \cap A' = \phi$

Laws of Algebra of Sets

If A, B and C are any three sets, then following are some fundamental laws of algebra of sets

(i) $A \cup A = A, A \cap A = A$ [Idempotent laws]
(ii) $A \cup \phi = A, A \cap U = A$ [Identity laws]
(iii) $A \cup B = B \cup A, A \cap B = B \cap A$ [Commutative laws]
(iv) $(A \cup B) \cup C = A \cup (B \cup C)$ [Associative laws]
$(A \cap B) \cap C = A \cap (B \cap C)$
(v) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ [Distributive laws]
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
(vi) $(A \cup B)' = A' \cap B'$ [De-Morgan's laws]
$(A \cap B)' = A' \cup B'$
(vii) $A - B = A \cap B', B - A = B \cap A'$
(viii) $(A - B) \cup B = A \cup B, (A - B) \cap B = \phi$
(ix) $A - (B \cap C) = (A - B) \cup (A - C)$
(x) $A - (B \cup C) = (A - B) \cap (A - C)$
(xi) $A \cap (B - C) = (A \cap B) - (A \cap C)$
(xii) $A \cap (B\Delta C) = (A \cap B)\Delta(A \cap C)$

Some Important Results

If A, B and C are finite sets and U be the finite universal set, then

(i) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
(ii) $n(A \cup B) = n(A) + n(B)$
$\Leftrightarrow A, B$ are disjoint non-void sets.
(iii) $n(A - B) = n(A) - n(A \cap B)$,
i.e. $n(A - B) + n(A \cap B) = n(A)$
(iv) $n(A \Delta B)$
= Number of elements which belong to exactly one of A or B
$= n[(A - B) \cup (B - A)]$
$= n(A - B) + n(B - A)$
[$\because (A - B)$ and $(B - A)$ are disjoint]
$= n(A) - n(A \cap B) + n(B) - n(A \cap B)$
$= n(A) + n(B) - 2n(A \cap B)$
(v) $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B)$
$- n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$
(vi) Number of elements is exactly two of the sets A, B, C
$= n(A \cap B) - 3n(A \cap B \cap C)$
$+ n(B \cap C) + n(C \cap A)$
(vii) Number of elements in exactly one of the sets A, B, C
$= n(A) + n(B) + n(C) - 2n(A \cap B) - 2n(B \cap C)$
$- 2n(A \cap C) + 3n(A \cap B \cap C)$
(viii) $n(A' \cup B') = n(A \cap B)' = n(U) - n(A \cap B)$
(ix) $n(A' \cap B') = n(A \cup B)' = n(U) - n(A \cup B)$

MASTER STROKES

1. If $X = \{8^n - 7n - 1 : n \in N\}$ and $Y = \{49(n - 1) : n \in N\}$, then
(a) $X \subset Y$
(b) $Y \subset X$
(c) $X = Y$
(d) None of the above

2. If A, B and C are three sets such that $A \cup B = A \cup C$ and $A \cap B = A \cap C$, then
(a) $A = B$ *(b)* $B = C$
(c) $A = C$ *(d)* $A = B = C$

3. Let A and B be two sets such that $A \cap X = B \cap X = \phi$ and $A \cup X = B \cup X$ for some set X. Then,
(a) $A = B$ *(b)* $A = X$ *(c)* $B = X$ *(d)* $A \cup B = X$

4. Let A and B be two sets defined as given below
$A = \{(x, y) : |x - 3| < 1 \text{ and } |y - 3| - 1\}$
$B = \{(x, y) : 4x^2 + 9y^2 - 32x - 54y + 109 \le 0\}$
Then,
(a) $A \subset B$ *(b)* $B \subset A$
(c) $A = B$ *(d)* None of these

5. If $A = \{(x, y) : x^2 + y^2 \le 1;\ x, y \in R\}$ and $B = \{(x, y) : x^2 + y^2 \le 4;\ x, y \in R\}$, then
(a) $A - B = A$ (b) $B - A = B$
(c) $A - B = \phi$ (d) $B - A = \phi$

6. If $A = \{\theta : 2\cos^2\theta + \sin\theta \le 2\}$ and $B = \left\{\theta : \frac{\pi}{2} \le \theta \le \frac{3\pi}{2}\right\}$, then $A \cap B$ is equal to
(a) $\left\{\theta : \frac{\pi}{2} \le \theta \le \frac{5\pi}{6}\right\}$
(b) $\left\{\theta : \pi \le \theta \le \frac{3\pi}{2}\right\}$
(c) $\left\{\theta : \frac{\pi}{2} \le \theta \le \frac{5\pi}{6}\right\} \cup \left\{\theta : \pi \le \theta \le \frac{3\pi}{2}\right\}$
(d) None of the above

7. In a certain town 25% families own a phone and 15% own a car, 65% families own neither a phone nor a car. 2000 families own both a car and a phone.
Consider the following statements in this regard
I. 10% families own both a car and a phone.
II. 35% families own either a car or a phone.
III. 40000 families live in the town.
Which of the following statements are correct?
(a) I and II (b) I and III
(c) II and III (d) I, II and III

8. Three sets A, B, C are such that $A = B \cap C$ and $B = C \cap A$, then
(a) $A \subset B$ (b) $A \supset B$
(c) $A = B$ (d) $A \subset B'$

9. If $A = \{x, y\}$, then the power set of A is
(a) $\{x^y, y^x\}$ (b) $\{\phi, x, y\}$
(c) $\{\phi, \{x\}, \{2y\}\}$ (d) $\{\phi, \{x\}, \{y\}, \{x, y\}\}$

10. The intersection of all the intervals having the form $\left[1 + \frac{1}{n}, 6 - \frac{2}{n}\right]$, where n is a positive integer, is
(a) [1, 6] (b) (1, 6)
(c) [2, 4] (d) [3/2, 5]

11. For any three sets A, B and C, the set $(A \cup B \cup C) \cap (A \cap B' \cap C') \cap C'$ is equal to
(a) $B \cap C'$ (b) $B' \cap C'$
(c) $B \cap C$ (d) $A \cap B \cap C$

12. Let $P = \{\theta : \sin\theta - \cos\theta = \sqrt{2}\cos\theta\}$ and $Q = \{\theta : \sin\theta + \cos\theta = \sqrt{2}\sin\theta\}$ be two sets. Then,
(a) $P \subset Q$ and $Q - P = \phi$ (b) $Q \not\subset P$
(c) $P \not\subset Q$ (d) $P = Q$

13. Let A and B be two sets containing 2 elements and 4 elements, respectively. The number of subsets of $A \times B$ having 3 or more elements is
(a) 256 (b) 220 (c) 219 (d) 211

14. If $A_1 \subset A_2 \subset A_3 \subset \ldots \subset A_{50}$ and $n(A_i) = i - 1$, then $n\left(\bigcap_{i=11}^{50} A_i\right)$ is equal to
(a) 49 (b) 50 (c) 11 (d) 10

15. If $A = \left\{p : p = \frac{(n+2)(2n^5 + 3n^4 + 4n^3 + 5n^2 + 6)}{n^2 + 2n},\ n, p \in Z^+\right\}$, then the number of elements in the set A, is
(a) 2 (b) 3 (c) 4 (d) 6

Answers

1. (a)	**2.** (b)	**3.** (a)	**4.** (a)	**5.** (c)
6. (c)	**7.** (c)	**8.** (c)	**9.** (d)	**10.** (c)
11. (a)	**12.** (d)	**13.** (c)	**14.** (d)	**15.** (c)

RELATIONS

A subset of cartesian product of sets is known as relation.

Cartesian Product of Sets

If A and B are any two non-empty sets, then the set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$ is called the cartesian product of the set A with set B and is denoted by $A \times B$. Thus, $A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$

Some Useful Results

(i) If $A = \phi$ or $B = \phi$, then $A \times B$ is defined as the null set.
(ii) In general $A \times B \neq B \times A$ and $A \times B = B \times A \Leftrightarrow A = B$
(iii) $A \times (B \cup C) = (A \times B) \cup (A \times C)$
(iv) $A \times (B \cap C) = (A \times B) \cap (A \times C)$
(v) $A \times (B - C) = (A \times B) - (A \times C)$
(vi) If $A \subseteq B$, then $A \times A \subseteq (A \times B) \cap (B \times A)$ and $A \times C \subseteq B \times C$ for any set C
(vii) If $A \subseteq B$ and $C \subseteq D$, then $A \times C \subseteq B \times D$
(viii) $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$
(ix) $(A \times B) \cap (B \times A) = (A \cap B) \times (B \cap A)$
(x) $A \times (B' \cup C')' = (A \times B) \cap (A \times C)$
(xi) $A \times (B' \cap C')' = (A \times B) \cup (A \times C)$
(xii) If A and B have n elements in common, then $A \times B$ and $B \times A$ have n^2 elements in common.

Relations

Let A and B be two sets. Then, a relation R from A to B is a subset of $A \times B$. If $(a, b) \in R$, then we write aRb which is read as a is related to b by the relation R. If $(a, b) \notin R$, then we write $a \not R\, b$ and we say that a is not related to b by the relation R.

Note If A and B are two finite sets consisting of m and n elements respectively, then $A \times B$ consists of mn ordered pairs. So, total number of subsets of $A \times B$ is 2^{mn}. Hence, 2^{mn} relations can be defined from A to B.

Domain and Range

If R is a relation from a set A to a set B, then the set of all first components or coordinates of the ordered pairs belonging to R is called the domain of R, while the set of all second components or coordinates of the ordered pairs in R is called the range of R.

Thus, Domain $(R) = \{a : (a, b) \in R\}$

and Range $(R) = \{b : (a, b) \in R\}$

Inverse Relation

Let A, B be two sets and let R be a relation from A to B. Then, the inverse of R, denoted by R^{-1}, is a relation from B to A, obtained from R by interchanging the components of ordered pairs in R.

Thus, $R^{-1} = \{(b, a) : (a, b) \in R\}$

Note Domain (R) = Range (R^{-1}) and Range (R) = Domain (R^{-1})

Types of Relations

Following are the types of relations

1. **Void relation** Let A be a non-empty set. Then, $\phi \subseteq A \times A$ and so it is a relation on set A. This relation is called the void or empty relation on set A.
2. **Universal relation** Let A be a non-empty set. Then, $A \times A$ is known as the universal relation on set A.
3. **Identity relation** Let A be a non-empty set. Then, $I_A = \{(a, a) : a \in A\}$ is called the identity relation on A. A relation I_A on a set A is called the identity relation, if every element of A is related to itself only.
4. **Reflexive relation** A relation R on a set A is said to be reflexive, if every element of A is related to itself. Thus, R is reflexive $\Leftrightarrow (a, a) \in R, \forall\, a \in A$. The identity relation is always reflexive. But a reflexive relation need not to be the identity relation.
5. **Symmetric relation** A relation R on a set A is said to be a symmetric relation iff $(a, b) \in R \Rightarrow (b, a) \in R, \forall\, a, b \in A$.

Note A relation R on a set A is symmetric iff $R = R^{-1}$. Also, the identity and universal relations on a set A are always symmetric.

6. **Transitive relation** A relation R on a set A is said to be a transitive relation iff $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R, \forall\, a, b, c \in A$.
7. **Anti-symmetric relation** A relation R on set A is said to be an anti-symmetric relation iff $(a, b) \in R$ and $(b, a) \in R \Rightarrow a = b, \forall\, a, b \in A$.
8. **Equivalence relation** A relation R on a set A is said to be an equivalence relation on A iff it is reflexive, symmetric and transitive.

Some Useful Results on Relations

Following are some useful results on relations

(i) The intersection of two equivalence relations on a set is an equivalence relation on the set.

(ii) The union of two equivalence relations on a set is not necessarily an equivalence relation on the set.

(iii) The inverse of an equivalence relation on a set is also an equivalence relation on the set.

(iv) A relation R on a set A is symmetric iff $R = R^{-1}$.

MASTER STROKES

1. Which of the following relations is not symmetric?
(a) R_1 on R defined by $(x, y) \in R_1 \Leftrightarrow 1 + xy > 0, \forall\, x, y \in R$
(b) R_2 on $N \times N$ defined by $(a, b) R_2 (c, d) \Leftrightarrow a + d = b + c, \forall\, a, b, c, d \in N$
(c) R_3 on Z defined by $(a, b) \in R_3 \Leftrightarrow b - a$ is an even integer
(d) R_4 on power set of a set X defined by $A\ R_4\ B$ iff $A \subseteq B$

2. Let A and B be two non-empty sets having n elements in common. Then, the number of elements common to $A \times B$ and $B \times A$ is
(a) $2n$ *(b)* n
(c) n^2 *(d)* None of these

3. Let A and B be two sets having 3 elements in common. If $n(A) = 5$ and $n(B) = 4$, then $n[(A \times B) \cap (B \times A)]$ is equal to
(a) 20 *(b)* 16 *(c)* 3 *(d)* 9

4. If $n(A \times B) = 45$, then $n(A)$ cannot be
(a) 15 *(b)* 17 *(c)* 5 *(d)* 9

5. If a relation R is defined on the set Z of integers as follows

$$(a, b) \in R \Leftrightarrow a^2 + b^2 = 25$$

Then, domain (R) is equal to
(a) $\{3, 4, 5\}$ *(b)* $\{0, 3, 4, 5\}$
(c) $\{0, \pm 3, \pm 4, \pm 5\}$ *(d)* None of these

6. Let $A = \{2, 3, 4, 5, \ldots, 17, 18\}$. Let '$\simeq$' be the equivalence relation on $A \times A$, cartesian product of A with itself, defined by $(a, b) \simeq (c, d)$ iff $ad = bc$. Then, the number of ordered pairs of the equivalence class of $(3, 2)$ is
(a) 4 *(b)* 5 *(c)* 6 *(d)* 7

7. Let R be the real line. Consider the following subsets of the plane $R \times R$.

$S = \{(x, y) : y = x + 1 \text{ and } 0 < x < 2\}$

$T = \{(x, y) : x - y \text{ is an integer}\}$

Which one of the following is true?

(a) S is an equivalence relation on R but T is not
(b) T is an equivalence relation on R but S is not
(c) Neither S nor T is an equivalence relation on R
(d) Both S and T are equivalence relations on R

8. The relation on the set $A = \{x : |x| < 3, x \in Z\}$ is defined by $R = \{(x, y) : y = |x|, x \neq -1\}$. Then, the number of elements in the power set of R is

(a) 32 *(b)* 16 *(c)* 8 *(d)* 64

9. The number of equivalence relations that can be defined on set $\{a, b, c\}$ is

(a) 3 *(b)* 5 *(c)* 7 *(d)* 8

10. Let a relation R in the set N of natural number be defined by $(x, y) \in R$ iff $x^2 - 4xy + 3y^2 = 0, \forall\, x, y \in N$. The relation R is

(a) reflexive and transitive *(b)* reflexive symmetric
(c) symmetric and transitive *(d)* on equivalence relation

11. If the number of different reflexive relations on a set A is equal to the number of different symmetric relations on set A, then the number of elements in A is

(a) 1 *(b)* 3
(c) 1 and 3 *(d)* 3 and 17

12. Consider the following relations

$R = \{(x, y) : x, y$ are real numbers and $x = wy$ for some rational number $w\}$

$S = \{m / n, p / q) : m, n, p$ and q are integers such that $n, q \neq 0$ and $qm = pn\}$

Then,

(a) S is an equivalence relation but R is not an equivalence relation
(b) R and S both are equivalence relations
(c) R is an equivalence relation but S is not equivalence relation
(d) neither R nor S is an equivalence relation

13. Let A and B be two sets containing four and two elements, respectively. Then, the number of subsets of the set $A \times B$, each having atleast three elements is

(a) 275 *(b)* 510 *(c)* 219 *(d)* 256

14. Let N be the set of natural numbers and for $a \in N$, aN denotes the set $\{ax : x \in N\}$. If $bN \cap cN = dN$, where b, c, d are natural numbers greater than 1 and the greatest common divisor (GCD) of b and c is 1, then d equals

(a) $b + c$ *(b)* $\{b, c\}$ *(c)* $\min\{b, c\}$ *(d)* bc

Answers

1. (d)	**2.** (c)	**3.** (d)	**4.** (b)	**5.** (c)
6. (c)	**7.** (b)	**8.** (b)	**9.** (b)	**10.** (a)
11. (b)	**12.** (a)	**13.** (c)	**14.** (d)	

FUNCTIONS

Let A and B be two non-empty sets. A relation from A to B is said to be a function from A to B, if

(i) for each $a \in A$, there exists $b \in B$ such that $(a, b) \in f$.
(ii) $(a, b) \in f$ and $(a, c) \in f \Rightarrow b = c$

Image, Pre-image, Domain, Co-domain, Range

If an element $a \in A$ is associated to an element $b \in B$, then b is called 'the f-image of a' or 'image of a under f' or 'the value of the function f at a'. Also, a is called the pre-image of b under the function f. We write it as $b = f(a)$.

If $f : A \to B$, then the set A is known as its domain and B is known as its co-domain. The set of all f-image of elements of A is known as the range of f or image set of A under f and is denoted by $f(A)$.

So, $f(A) = \{f(x) : x \in A\}$ = Range of f and $f(A) \subseteq B$.

Equal Functions

Two functions $f : A \to B$ and $g : C \to D$ are said to be equal iff

(i) $A = C$, i.e. Domain of f = Domain of g
(ii) $B = D$, i.e. Co-domain of f = Co-domain of g
(iii) $f(x) = g(x)$ for every x belonging to their common domain.

Number of Functions

Let A and B be two finite sets having m and n elements, respectively. Then, the total number of functions from A to B is n^m.

Types of Functions

Following are types of functions

1. One-one Function

A function $f : A \to B$ is said to be a one-one function or an injection, if different elements of A have different images in B. Thus, $f : A \to B$ is one-one.

$\Leftrightarrow \quad a \neq b \Rightarrow f(a) \neq f(b), \forall\, a, b \in A$

$\Leftrightarrow \quad f(a) = f(b) \Rightarrow a = b, \forall\, a, b \in A$

Note
- If $f : A \to B$ is not one-one, then it is said to be a many-one function.
- If A and B are finite sets having m and n elements, respectively. Then, number of one-one functions from A to B

$$= \begin{cases} {}^nC_m \times m!, & \text{if } m \leq n \\ 0, & \text{if } m > n \end{cases}$$

2. Onto Function

A function $f : A \to B$ is said to be an onto function or a surjection, if every element of B is the f-image of some element of A, i.e. if $f(A) = B$ or range of f is the co-domain of f.

Note
- If $f : A \to B$ is not an onto function, then it is called into function.
- If A and B are two sets having m and n elements, respectively such that $1 \leq n \leq m$, then the number of onto functions from A to B is equal to $= \sum_{r=1}^{n} (-1)^{n-r}\, {}^nC_r r^m$

3. Bijection (One-one onto function)

A function $f : A \to B$ is called a bijection, if it is one-one as well as onto both.

Composition of Functions

Let $f : A \to B$ and $g : C \to D$ be two functions such that range $(f) \subseteq C$ (domain g), then corresponding to each $x \in A$, there exists unique element $f(x) \in C$, whose image under g, i.e. $g(f(x))$ is a unique elements in D. Thus, for each $x \in A$, there exists a unique element $g(f(x)) \in D$. This gives a function from A to D which associates each $x \in A$ to $g(f(x)) \in D$. This function is known as the composition of f and g and is denoted by gof.

Properties of Composition of Functions

Following are the properties of composition of function

Property 1 The composition of function is not commutative, i.e. $fog \neq gof$.

Property 2 The composition of function is associative, i.e. $(fog)oh = fo(goh)$.

Property 3 The composition of two bijections is a bijection.

Property 4 Let $f : A \to B$. Then, $foI_A = I_B of = f$, where I_A or I_B are identity function.

Property 5 Let $f : A \to B$, $g : B \to A$ be two functions such that $gof = I_A$. Then, f is an injection and g is a surjection.

Property 6 Let $f : A \to B$ and $g : B \to A$ be two functions such that $fog = I_B$. Then, f is a surjection and g is a injection.

Property 7 Let $f : A \to B$ and $g : B \to C$ be two functions. Then,

(i) $gof : A \to C$ is onto $\Rightarrow y : B \to C$ is onto.

(ii) $gof : A \to C$ is one-one $\Rightarrow f : A \to B$ is one-one.

(iii) $gof : A \to C$ is onto and $g : B \to C$ is one-one $\Rightarrow f : A \to B$ is onto.

(iv) $gof : A \to C$ is one-one and $f : A \to B$ is onto $\Rightarrow g : B \to C$ is one-one.

Inverse of a Function

Let $f : A \to B$ be a bijection. Then, a function $g : B \to A$ which associates each element $y \in B$ to a unique element $x \in A$ such that $f(x) = y$ is called the inverse of f.

i.e. $\quad f(x) = y \Leftrightarrow g(y) = x$

The inverse of f is generally denoted by f^{-1}.

Thus, if $f : A \to B$ is a bijection, then $f^{-1} : B \to A$ is such that $\quad f(x) = y \Leftrightarrow f^{-1}(y) = x$.

Properties of Inverse of a Function

Property 1 The inverse of a bijection is unique.

Property 2 The inverse of a bijection is also a bijection.

Property 3 If $f : A \to B$ is a bijection and $g : B \to A$ is the inverse of f, then $fog = I_B$ and $gof = I_A$, where I_A and I_B are the identity functions on the sets A and B, respectively.

Note If in the above property, we have $B = A$. Then, we find that for every bijection $f : A \to A$ there exists a bijection $g : A \to A$ such that $fog = gof = I_A$

Property 4 If $f : A \to B$ and $g : B \to C$ are two bijections, then $gof : A \to C$ is a bijection and $(gof)^{-1} = f^{-1}og^{-1}$.

Property 5 Let $f : A \to B$ and $g : B \to A$ be two functions such that $gof = I_A$ and $fog = I_B$. Then, f and g are bijections and $g = f^{-1}$.

Method of Finding the Inverse of a Bijection

Method I

***Step* I** Obtain the bijection, let it be $f : A \to B$.

***Step* II** Put $f(x) = y$, where $y \in B$ and $x \in A$.

***Step* III** Solve $f(x) = y$ to obtain x in terms of y.

***Step* IV** Replace x by $f^{-1}(y)$ in the relation obtained in step III to obtain the inverse of f.

Method II

***Step* I** Obtain the bijection $f(x)$.

***Step* II** In the formula for $f(x)$, replace $f(x)$ by x and x by $f^{-1}(x)$.

***Step* III** Simplify the relation obtained in step II to get $f^{-1}(x)$.

MASTER STROKES

1. If $f(x)=\cos(\log x)$, then $f(x)\,f(y)-\frac{1}{2}\left\{f\left(\frac{x}{y}\right)+f(xy)\right\}$ is equal to

(a) 0 (b) $\frac{1}{2}f(x)\,f(y)$

(c) $f(x+y)$ (d) None of these

2. If functions f and g given by

$f(x)=\log(x-1)-\log(x-2)$ and $g(x)=\log\left(\frac{x-1}{x-2}\right)$

are equal, then x lies in the interval

(a) $[1,2]$ (b) $[2,\infty)$

(c) $(2,\infty)$ (d) $(-\infty,\infty)$

3. If $P=\{a,b,c\}$ and $Q=\{1,2\}$, then the total number of relations from P to Q, which are not functions, is

(a) 56 (b) 8

(c) 9 (d) 55

4. The function $f:R\to R$ defined by $f(x)=2^x+2^{|x|}$ is

(a) one-one and onto

(b) many-one and onto

(c) one-one and into

(d) many-one and into

5. The set of values of parameter 'a' for which the function $f:R\to R$ defined by $f(x)=ax+\sin x$ is bijective is

(a) $[-1,1]$ (b) $R-(-1,1)$

(c) $R-[-1,1]$ (d) $(-1,1)$

6. If $g(x)=x^2+x-2$ and $\frac{1}{2}gof(x)=2x^2-5x+2$, then $f(x)$ is equal to

(a) $2x-3$ (b) $2x+3$

(c) $2x^2+3x+1$ (d) $2x^2-3x-1$

7. If $f(x)=\frac{1-x}{1+x}$, $x\neq -1$, then $f^{-1}(x)$ is equal to

(a) $f(x)$ (b) $\frac{1}{f(x)}$

(c) $-f(x)$ (d) $-\frac{1}{f(x)}$

8. Let $f(x)=\frac{1}{1-x}$. Then, $\{fo(fof)\}(x)$

(a) $x,\ \forall\, x\in R$ (b) $x,\ \forall\, x\in R-\{1\}$

(c) $x,\ \forall\, x\in R-\{0,1\}$ (d) None of these

9. Suppose $f(x)=(x+1)^2$, $\forall\ x\geq -1$. If $g(x)$ is the function whose graph is the reflection of the graph of $f(x)$ with respect to the line $y=x$, then $g(x)$ equals

(a) $-\sqrt{x}-1,\ x\geq 0$ (b) $\frac{1}{(x+1)^2},\ x>-1$

(c) $\sqrt{x+1},\ x\geq -1$ (d) $\sqrt{x}-1,\ x\geq 0$

10. If $f:\{1,2,3,4\}\to\{1,4,9,16\}$ and $g:\{1,4,9,16\}\to\left\{1,\frac{1}{2},\frac{1}{3},\frac{1}{4}\right\}$ are two bijective functions such that $x_1>x_2\Rightarrow f(x_1)<f(x_2)$, $g(x_1)>g(x_2)$, then $f^{-1}\left(g^{-1}\left(\frac{1}{2}\right)\right)$ is equal to

(a) 1 (b) 4

(c) 16 (d) None of these

11. For a real number x, let $[x]$ denotes the greatest integer less than or equal to x. Let $f:R\to R$ be defined by $f(x)=2x+[x]+\sin x\cos x$. Then, f is

(a) one-one but not onto (b) onto but not one-one

(c) both one-one and onto (d) neither one-one nor onto

12. Let $f:(0,1)\to R$ be defined by $f(x)=\frac{b-x}{1-bx}$, where b is a constant such that $0<b<1$. Then,

(a) f is not invertible on (0, 1)

(b) $f\neq f^{-1}$ on (0, 1) and $f'(b)=\frac{1}{f'(0)}$

(c) $f=f^{-1}$ on (0, 1) and $f'(b)=\frac{1}{f'(0)}$

(d) f^{-1} is differentiable on (0, 1)

13. If $f(x)=\frac{3x+2}{5x-3}$, then

(a) $f^{-1}(x)=f(x)$

(b) $f^{-1}(x)=-f(x)$

(c) $(fof)(x)=-x$

(d) $f^{-1}(x)=\frac{-1}{19}f(x)$

14. Given, $f(x)=\log\left(\frac{1+x}{1-x}\right)$ and $g(x)=\frac{3x+x^3}{1+3x^2}$, then $fog(x)$ equals

(a) $-f(x)$ (b) $3f(x)$

(c) $[f(x)]^3$ (d) None of these

Answers

1. (a)	2. (c)	3. (a)	4. (c)	5. (c)
6. (a)	7. (a)	8. (c)	9. (d)	10. (d)
11. (c)	12. (a)	13. (a)	14. (b)	

MATHEMATICAL REASONING

Mathematical reasoning is the critical skill that enables a student to make use of all other mathematical skills.

Mathematical Logic

1. **Assertive sentence** A sentence that makes an assertion is called an assertive sentence or a declarative sentence.
2. **Imperative sentence** A sentence that expresses a request or a command is called an imperative sentence.
3. **Exclamatory sentence** A sentence that expresses some strong feeling is called an exclamatory sentence.
4. **Interrogative sentence** A sentence that asks some question is called an interrogative sentence.

Statements or Propositions

A statement or a proposition is an assertive (or a declarative) sentence which is either true or false but not both.

1. **Open statement** A declarative sentence containing variable(s) is an open statement, if it becomes a statement when the variable(s) is(are) replaced by some definite value(s).
2. **Truth set** The truth or falsity of a statement is called its truth value. If a statement is true, then we say that its truth value is 'True' or 'T'. On the other hand the truth value of a false statement is 'False' or 'F'.
3. **Simple statement** Any statement or proposition whose truth value does not explicity depend on another statement is said to be a simple statement.
4. **Compound statements** If a statement is combination of two or more simple statements, then it is said to be a compound statement or a compound proposition.
5. **Connectives** The phrases or words which connect simple statements are called logical connectives or sentenial connectives or simply connectives or logical operators.

In the following table, we list some connectives, their symbol and the nature of the compound statement formed by them.

Connective	Symbol	Nature of the compound statement formed by using the connective
and	$\wedge$	Conjunction
or	$\vee$	Disjunction
If…then	$\Rightarrow$ or $\rightarrow$	Implication or conditional
If and only if (iff)	$\Leftrightarrow$ or $\leftrightarrow$	Equivalence or bio-conditional
not	$\sim$ or $\neg$	Negation

Truth Tables

A table that shows the relationship between the truth value of a compound statement $S(p, q, r, \ldots)$ and the truth value of its sub-statements $p, q, r, \ldots$ etc., is called the truth table of statements.

Basic Logical Connectives

Following are some basic logical connectives

1. Conjunction

Any two simple statements can be connected by the word "and" to form a compound statement called the conjunction of the original statements. Symbolically, if p and q are two simple statements, then $p \wedge q$ denotes the conjunctions of p and q, and is read as "p and q".

Truth table of $p \wedge q$ is as follow

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

2. Disjunction or Alternation

Any two statements can be connected by the word "or" to form a compound statement called the disjunction of the original statements. Symbolically, if p and q are two simple statements, then $p \vee q$ denotes the disjunction of p and q, and is read as "p or q".

Truth table of $p \vee q$ is as follow

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

3. Negation

The denial of a statement p is called its negation, written as $\sim p$. Negation of any statement p is formed by writing "It is not the case that …" or "It is false that …" before p or, if possible by inserting in p the word "not".

Truth table of $\sim p$ is as follow

p	$\sim p$
T	F
F	T

Conditional and Biconditional Statements

In mathematics, we come across many statements of the form "if p then q" and "p if and only if q" such statements are called conditional statements.

Implication or Conditional Statements

Two statements connected by the connective phrase 'if... then' give rise to a compound statement which is known as an implication or a conditional statement.

If p and q are two statements forming the implication 'if p then q', then we denote this implication by

"$p \Rightarrow q$" or "$p \to q$"

In the implication "$p \Rightarrow q$", p is called the antecedent or hypothesis and q the consequent or conclusion.

Truth table of $p \Rightarrow q$ is as follow

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Note The truth table of $p \Rightarrow q$ is same as that of $\sim p \vee q$

$$p \Rightarrow q \equiv \sim p \vee q$$

Biconditional Statement

A statement is a biconditional statement, if it is the conjunction of two conditional statements one converse to the other. Thus, if p and q are two statements, then the compound statement p if and only if q is called a biconditional statement or an equivalence and is denoted by $p \Leftrightarrow q$.

Thus, $p \Leftrightarrow q : (p \Rightarrow q) \wedge (q \Rightarrow p)$

Truth table of $p \Leftrightarrow q$ is as follow

p	q	$p \Rightarrow q$	$q \Rightarrow p$	$p \Leftrightarrow q = (p \Rightarrow q) \wedge (q \Rightarrow p)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Logical Equivalence

Two compound statements $S_1(p, q, r, \ldots)$ and $S_2(p, q, r, \ldots)$ are said to be logically equivalent, or simply equivalent, if they have the same truth values for all logically possibilities.

Negation of Compound Statements

Following are the rules of negation of compound statements

(i) $\sim(p \wedge q) \equiv \sim p \vee \sim q$

(ii) $\sim(p \vee q) \equiv \sim p \wedge \sim q$

(iii) $\sim(p \Rightarrow q) \equiv p \wedge \sim q$

(iv) $\sim(p \Leftrightarrow q) \equiv (p \wedge \sim q) \vee (q \wedge \sim p)$

Tautologies and Contradiction

A statement pattern is called a tautology, if it is always true, whatever may be the truth values of constitute statements. A statement pattern is called a contradiction, it is always false, whatever may be the truth values of constitute statements. A compound statement which is neither a tautology nor a contradiction is called a contigency.

Note The negation of a tautology is a contradiction and *vice-versa*.

Algebra of Statements

(i) **Idempotent laws** For any statement p, we have

(a) $p \vee p \equiv p$ (b) $p \wedge p \equiv p$

(ii) **Commutative laws** For any two statements p and q, we have

(a) $p \vee q \equiv q \vee p$ (b) $p \wedge q \equiv q \wedge p$

(iii) **Associative laws** For any three statements p, q, r we have

(a) $(p \vee q) \vee r \equiv p \vee (q \vee r)$

(b) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

(iv) **Distributive laws** For any three statements p, q, r we have

(a) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

(b) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

(v) **De-Morgan's laws** If p and q are two statements, then

(a) $\sim(p \wedge q) \equiv \sim p \vee \sim q$ (b) $\sim(p \vee q) \equiv \sim p \wedge \sim q$

(vi) **Identity laws** If t and c denote a tautology and a contradiction respectively, then for any statement p, we have

(a) $p \wedge t \equiv p$ (b) $p \vee c \equiv p$

(c) $p \vee t \equiv t$ (d) $p \wedge c \equiv c$

(vii) **Complement laws** For any statement p, we have

(a) $p \vee \sim p \equiv t$ (b) $p \wedge \sim p \equiv c$

(c) $\sim t \equiv c$ (d) $\sim c \equiv t$

(viii) **Law of contrapositive** For any two statements p and q, we have $p \Rightarrow q \equiv \sim q \Rightarrow \sim p$

(ix) **Involution laws** For any statement p, we have $\sim(\sim p) \equiv p$

Duality

Two compound statements S_1 and S_2 are said to be duals of each other, if one can be obtained from the other by replacing $\wedge$ by $\vee$ and $\vee$ by $\wedge$.

MASTER STROKES

1. Consider the following statements
p : I have the raincoat, q : I can walk in the rain.
The propositions "If I have the raincoat, then I can walk in the rain" is represented by
(a) $p \to q$ *(b)* $p \vee q$ *(c)* $p \wedge q$ *(d)* $p \leftrightarrow q$

2. The logically equivalent proposition of $p \leftrightarrow q$ is
(a) $(p \to q) \wedge (q \to p)$ *(b)* $(p \to q) \vee (q \to p)$
(c) $(p \wedge q) \to (p \vee q)$ *(d)* $(p \wedge q) \vee (p \vee q)$

3. If the inverse of implication $p \to q$ is defined as $\sim p \to \sim q$, then the inverse of the proposition $(p \wedge \sim q) \to r$ is
(a) $\sim r \to \sim p \vee q$ *(b)* $\sim p \vee q \to \sim r$
(c) $r \to p \wedge \sim q$ *(d)* None of these

4. Which of the following is logically equivalent to $(p \wedge q)$?
(a) $p \to \sim q$ *(b)* $\sim p \vee \sim q$
(c) $\sim (p \to \sim q)$ *(d)* $\sim (\sim p \wedge \sim q)$

5. $\sim p \vee \sim q$ is logically equivalent to
(a) $\sim p \to \sim q$ *(b)* $p \wedge q$ *(c)* $p \to \sim q$ *(d)* $p \leftrightarrow q$

6. If $p \to (q \vee r)$ is false, then the truth values of p, q, r are respectively
(a) T, F, F *(b)* F, F, F *(c)* F, T, T *(d)* T, T, F

7. $(p \wedge \sim q) \wedge (\sim p \vee q)$ is
(a) a tautology
(b) a contradiction
(c) both a tautology and a contradiction
(d) neither a tautology nor a contradiction

8. The proposition $(p \to \sim p) \wedge (\sim p \to p)$ is
(a) a tautology
(b) a contradiction
(c) neither a tautology nor a contradiction
(d) a tautology and a contradiction

9. Negation of the statement $(p \wedge r) \to (r \vee q)$ is
(a) $(p \wedge r) \wedge (\sim r \wedge \sim q)$ *(b)* $\sim (p \wedge r) \to \sim (r \vee q)$
(c) $\sim (p \vee r) \to \sim (r \wedge q)$ *(d)* $(p \wedge r) \vee (r \vee q)$

10. The contrapositive of $p \to (\sim q \to \sim r)$ is
(a) $(\sim q \wedge r) \to \sim p$ *(b)* $(q \wedge \sim r) \to \sim p$
(c) $p \to (\sim r \vee q)$ *(d)* $p \wedge (q \vee r)$

11. The statement $\sim (p \leftrightarrow \sim q)$ is
(a) a tautology *(b)* a fallacy
(c) equivalent to $p \leftrightarrow q$ *(d)* equivalent to $\sim p \leftrightarrow q$

12. The only statement among the followings is a tautology
(a) $A \vee (A \wedge B)$ *(b)* $[A \wedge (A \to B)] \to B$
(c) $B \to [A \wedge (A \to B)]$ *(d)* $A \wedge (A \vee B)$

13. The negation of $\sim s \vee (\sim r \wedge s)$ is equivalent to
(a) $s \vee (r \vee \sim s)$ *(b)* $s \wedge r$
(c) $s \wedge \sim r$ *(d)* $s \wedge (r \wedge \sim s)$

14. Let p and q be any two propositions.
Statement 1 $(p \to q) \leftrightarrow q \vee \sim p$ is a tautology.
Statement 2 $\sim (\sim p \wedge q) \wedge (p \vee q) \leftrightarrow p$ is a fallacy.
(a) Both Statements are true
(b) Both Statements are false
(c) Only Statement 1 is true
(d) Only Statement 2 is true

15. The statement $p \to (q \to p)$ is equivalent to
(a) $p \to (p \wedge q)$ *(b)* $p \to (p \leftrightarrow q)$
(c) $p \to (p \to q)$ *(d)* $p \to (p \vee q)$

Answers

1. (a)	**2.** (a)	**3.** (b)	**4.** (d)	**5.** (c)
6. (a)	**7.** (b)	**8.** (b)	**9.** (a)	**10.** (a)
11. (c)	**12.** (b)	**13.** (b)	**14.** (d)	**15.** (d)

MATHEMATICS IN action!

The Number Two (2)

Two is descended from the Greek root *dyo* and the Latin root *duo* through the Old English *twa*. Early languages often had both feminine and masculine forms for two and so there are abundant and diverse roots related to 'two-ness'. Many 'two' words use the Greek root *bi*; biannual, binary, biscuit and biceps are examples. Others come from the Old English *twa*, such as between, twilight, twist and twin. From *duo* we get dual, duet, dubious (of two minds), duplex (two layers) and double. The Latin *di* gives us diploma (two papers) and dihedral. The earlier Greek *dyo* produces dyad, composed of two parts. When *am* precedes *bi*, as in ambivalent, it means 'either of two'. The word *didymous* is Greek for twin and is used in scientific terms to represent things which occur in pairs. Students of the bible may remember that didymous is also the nickname of the disciple Thomas (John 11:16). Thomas itself is from the old Aramaic word for twin, *t'oma*, which was also used by the Greeks and later made its way into Latin.

In Henry B. Fine's *The Number System of Algebra*, 1903, he points out that the name for two often was drawn from dual body parts, such as eyes, ears, or wings. 'Two' in Chinese is a word meaning 'ears', in Thibet 'wing' in Hottnetot 'hand'. It would seem that such a system would be thwarted by the difficulty of finding body parts to represent three and four.

COMPLEX NUMBERS

A number of the form $x + iy$, where $x, y \in R$ and $i = \sqrt{-1}$, is called a complex number. If $z = x + iy$, then x is called real part of z and denoted as $\text{Re}(z)$, whereas y is called imaginary part of z and denoted as $\text{Im}(z)$.

Note Complex number may be defined as an ordered pair of real numbers written as (x, y). A complex number is purely real, if $\text{Im}(z) = 0$ and it is purely imaginary, if $\text{Re}(z) = 0$.

POINTS TO REMEMBER

- If $(a + ib)$ (< or >) $c + id$ is not defined, i.e. order relation in complex number is not defined (comparision of complex number is only defined when imaginary part is zero).
- $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ is valid. If atleast one of a and b is non-negative. If a and b both are negative, then $\sqrt{a} \times \sqrt{b} = -\sqrt{ab}$.

Integral Power of Iota (i)

As we know that square of every real number is positive, hence solution of the equation $x^2 = -9$ is not possible in real number system. $i = \sqrt{-1}$ is known as iota which is introduced in complex number system.

Also, $i = \sqrt{-1}$, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$

$i^{4n} = 1$, $i^{4n+1} = i$, $i^{4n+2} = -1$ and $i^{4n+3} = -i$

SPEED BOOSTER The sum of four consecutive powers of i is zero, i.e. $i^n + i^{n+1} + i^{n+2} + i^{n+3} = 0$.

Equality of Complex Number

Two complex numbers $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ are equal, if $a_1 = a_2$ and $b_1 = b_2$, i.e. $\text{Re}(z_1) = \text{Re}(z_2)$ and $\text{Im}(z_1) = \text{Im}(z_2)$

Fundamental Operations on Complex Numbers

Addition

Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ be two complex numbers. Then, their sum $z_1 + z_2$ is defined as the complex number $(a_1 + a_2) + i(b_1 + b_2)$.

Clearly, $\text{Re}(z_1 + z_2) = \text{Re}(z_1) + \text{Re}(z_2)$

and $\text{Im}(z_1 + z_2) = \text{Im}(z_1) + \text{Im}(z_2)$

Properties of Addition

(i) **Addition is commutative** For any two complex numbers z_1 and z_2, we have

$$z_1 + z_2 = z_2 + z_1$$

(ii) **Addition is associative** For any three complex numbers z_1, z_2 and z_3 we have

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$$

(iii) **Existence of additive identity** The complex number $0 = 0 + i\,0$ is the identity element for addition, i.e. $z + 0 = z = 0 + z, \forall\, z \in C$

(iv) **Existence of additive inverse** For every complex number z there exists $-z$ such that

$$z + (-z) = 0 = (-z) + z$$

The complex number $-z$ is called the additive inverse of z.

Subtraction

Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ be two complex numbers. Then, the subtraction of z_2 from z_1 is denoted by $z_1 - z_2$ and is defined as the addition of z_1 and $-z_2$.

Thus, $z_1 - z_2 = z_1 + (-z_2)$

$$= (a_1 + ib_1) + (-a_2 - ib_2)$$
$$= (a_1 - a_2) + i(b_1 - b_2)$$

Multiplication

Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ be two complex numbers. Then, the multiplication of z_1 with z_2 is denoted by $z_1 z_2$ and is defined as the complex number.

$$(a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1).$$

Thus, $z_1 z_2 = (a_1 + ib_1)(a_2 + ib_2)$

$\Rightarrow$ $z_1 z_2 = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)$

$\Rightarrow$ $z_1 z_2 = [\text{Re}(z_1)\text{Re}(z_2) - \text{Im}(z_1)\,\text{Im}(z_2)]$

$$+ i[\text{Re}(z_1)\,\text{Im}(z_2) + \text{Re}(z_2)\,\text{Im}(z_1)]$$

Properties of Multiplication

(i) **Multiplication is commutative** For any two complex numbers z_1 and z_2, we have

$$z_1 z_2 = z_2 z_1$$

(ii) **Multiplication is associative** For any three complex numbers z_1, z_2 and z_3, we have

$$(z_1 z_2) z_3 = z_1 (z_2 z_3)$$

(iii) **Existence of identity element for multiplication** The complex number $1 = 1 + i0$ is the identity element for multiplication, i.e. for every complex number z, we have

$$z \cdot 1 = z = 1 \cdot z$$

(iv) **Existence of multiplicative inverse** Corresponding to every non-zero complex number $z = a + ib$ there exists a complex number $z_1 = x + iy$ such that

$$z \cdot z_1 = 1 = z_1 \cdot z$$

The complex number z_1 is called the multiplicative inverse or reciprocal of z and is given by

$$z_1 = \frac{a}{a^2 + b^2} + \frac{i(-b)}{a^2 + b^2}$$

(v) **Multiplication of complex numbers is distributive over addition of complex numbers** For any three complex numbers z_1, z_2 and z_3, we have

(i) $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$ [left distributivity]

(ii) $(z_2 + z_3) z_1 = z_2 z_1 + z_3 z_1$ [right distributivity]

Division

The division of a complex number z_1 by a non-zero complex number z_2 is defined as the multiplication of z_1 by the multiplicative inverse of z_2 and is denoted by $\frac{z_1}{z_2}$.

Thus, $$\frac{z_1}{z_2} = z_1 \cdot z_2^{-1} = z_1\left(\frac{1}{z_2}\right)$$

Conjugate of Complex Number

Complex numbers $z = a + ib$ and $\bar{z} = a - ib$ are called conjugate to each other (here, complex conjugate are obtained by just changing the sign of i).

Also, $$\text{Re}(z) = \frac{z + \bar{z}}{2} = a$$

$$\text{Im}(z) = \frac{z - \bar{z}}{2i} = b$$

Properties of Conjugate of a Complex Number

If z is a complex number, then following are the properties of conjugate of complex number

(i) $\overline{(\bar{z})} = z$

(ii) $z + \bar{z} = 2\,\text{Re}(z),\ z - \bar{z} = 2i\,\text{Im}(z)$

(iii) $z + \bar{z} = 0 \Leftrightarrow z$ is purely imaginary; $z - \bar{z} = 0 \Leftrightarrow z$ is purely real.

(iv) $z\bar{z} = [\text{Re}(z)]^2 + [\text{Im}(z)]^2$

(v) $\overline{(z_1 + z_2)} = \bar{z}_1 + \bar{z}_2$

(vi) $\overline{(z_1 - z_2)} = \bar{z}_1 - \bar{z}_2$

(vii) $\overline{(z_1 z_2)} = \bar{z}_1 \bar{z}_2$

(viii) $\left(\overline{\frac{z_1}{z_2}}\right) = \frac{\bar{z}_1}{\bar{z}_2};\quad z_2 \neq 0$

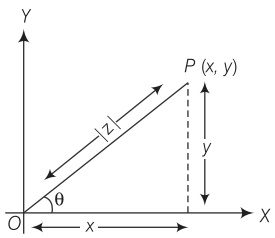
Modulus and Argument of Complex Number

If complex number $z = x + iy$, then modulus of complex number z is $|z|$, which is the distance of point 'z' from the origin and inclination of OP with positive X-axis is called argument of the complex number. Hence,

$$|z| = \sqrt{x^2 + y^2}$$

and $$\arg(z) = \tan^{-1}\left|\frac{y}{x}\right| + 2k\pi,\ k \in I$$

Note The complex number z represents a point, where as $|z|$ is distance of this point from origin.



Properties of Modulus

(i) $|z| = 0 \Leftrightarrow z = 0$ (ii) $|z| = |\bar{z}| = |-z| = |-\bar{z}|$

(iii) $|z_1 z_2| = |z_1||z_2|$ (iv) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, if $z_2 \neq 0$

(v) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + \bar{z}_1 z_2 + z_1 \bar{z}_2$
$= |z_1|^2 + |z_2|^2 + 2\,\text{Re}(z_1 \bar{z}_2)$

(vi) $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2\,\text{Re}(z_1 \bar{z}_2)$

(vii) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$

(viii) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2|z_1||z_2|\cos(\theta_1 - \theta_2)$

(ix) $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2|z_1||z_2|\cos(\theta_1 - \theta_2)$

Properties of Argument

(i) $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2) + 2k\pi,\ k = 0 \text{ or } 1 \text{ or } -1$

(ii) $\arg\left(\frac{z}{\bar{z}}\right) = 2\arg(z) + 2k\pi,\ k = 0 \text{ or } 1 \text{ or } -1$

(iii) $\arg(z^n) = n\arg(z) + 2k\pi,\ k = 0,\ -1 \text{ or } 1$

(iv) If $\arg\left(\frac{z_2}{z_1}\right) = \theta$, then $\arg\left(\frac{z_1}{z_2}\right) = 2k\pi - \theta$, where $k \in I$

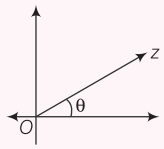
(v) $\arg(\bar{z}) = -\arg(z)$

(vi) If $\arg(z) = 0 \Rightarrow z$ is real.

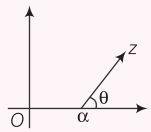
(vii) $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) + 2k\pi,\ k = 0 \text{ or } -1 \text{ or } 1$

Facts Based on Arguments

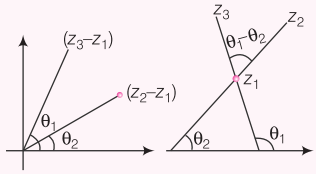
- Locus of z, if $\arg(z) = \theta$ is ray excluding origin.



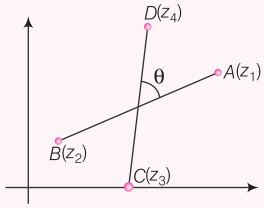
- Locus of z, if $\arg(z - \alpha) = \theta$ (fixed) and $\alpha > 0$ is a ray.



- Angle between two lines $= \theta_1 - \theta_2$



- Angle between line joining z_1, z_2, z_3 and z_4 is $\theta = \arg\left(\frac{z_4 - z_3}{z_2 - z_1}\right)$



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If $|z| = a$, then maximum and minimum values of $|z + c + id|$ is $(a + \sqrt{c^2 + d^2})$ and $(a - \sqrt{c^2 + d^2})$.

Trigonometrical or Polar Form of a Complex Number

Let $z = a + ib$, then the polar form of z is $r(\cos\theta + i\sin\theta)$, i.e. $a + ib = r(\cos\theta + i\sin\theta)$, where

$$r = |z| \text{ and } \theta = \tan^{-1}\left|\frac{b}{a}\right| + 2k\pi,\ k \in I$$

Note Since, $e^{i\theta} = \cos\theta + i\sin\theta$

$\therefore$ $a + ib = r(\cos\theta + i\sin\theta) \Rightarrow a + ib = r \cdot e^{i\theta}$

This is the Euler's form of the complex numbers.

Product and Division of Two Complex Numbers

Let two complex numbers be $z_1 = e^{i\theta}$ and $z_2 = e^{i\phi}$

Now, $z_1 . z_2 = |z_1| e^{i\theta} \times |z_2| e^{i\phi}$

$z_1 . z_2 = |z_1||z_2| e^{i(\theta + \phi)}$

$z_1 . z_2 = |z_1||z_2|[\cos(\theta + \phi) + i\sin(\theta + \phi)]$

and $\frac{z_1}{z_2} = \frac{|z_1| e^{i\theta}}{|z_2| e^{i\phi}} = \frac{|z_1|}{|z_2|} e^{i(\theta - \phi)}$

$= \frac{|z_1|}{|z_2|}[\cos(\theta - \phi) + i\sin(\theta - \phi)]$

Logarithm of a Complex Number

Let $z = a + ib$, then $\log z = \log(a + ib) = \log(|z| e^{i\theta})$

$[\because a + ib = |z| e^{i\theta}]$

$\Rightarrow \log z = \log|z| + \log e^{i\theta} \Rightarrow \log z = \log|z| + i\theta$

$\Rightarrow \log z = \log|z| + i \arg(z)$

Square Roots of a Complex Number

Let $z = a + ib$, then square root of z

i.e. $\sqrt{z} = \pm\left[\sqrt{\frac{|z| + a}{2}} + i\sqrt{\frac{|z| - a}{2}}\right]$ for $b > 0$

$= \pm\left[\sqrt{\frac{|z| + a}{2}} - i\sqrt{\frac{|z| - a}{2}}\right]$ for $b < 0$

POINTS TO REMEMBER

- The square roots of $i = \pm\left[\frac{1 + i}{\sqrt{2}}\right]$
- The square roots of $-i = \pm\left[\frac{1 - i}{\sqrt{2}}\right]$

Cube Roots of Unity

Let $\sqrt[3]{1} = z \Rightarrow z^3 - 1 = 0 \Rightarrow (z - 1)(z^2 + z + 1) = 0$

$\therefore$ $z = 1, \frac{-1 + i\sqrt{3}}{2}, \frac{-1 - i\sqrt{3}}{2}$

Note If second root represented by ω, then third root is ω^2. Therefore, cube roots of unity are $1, \omega, \omega^2$.

Properties of Cube Roots of Unity

(i) $1 + \omega + \omega^2 = 0$

(ii) $\omega^3 = 1$

(iii) $\omega^{3n} = 1, \omega^{3n+1} = \omega, \omega^{3n+2} = \omega^2$

(iv) $\bar{\omega} = \omega^2$ and $(\bar{\omega})^2 = \omega$

(v) Each complex cube root of unity is the square of the other.

(vi) Each complex cube root of unity is the reciprocal of the other. i.e. $\omega = \frac{1}{\omega^2}$ and $\omega^2 = \frac{1}{\omega}$

(vii) Cube roots of -1 are $-1, -\omega$ and $-\omega^2$.

(viii) Cube roots of unity represent vertices of equilateral triangle on the argand plane.

IMPORTANT POINTS

- $x^2 + x + 1 = (x - \omega)(x - \omega^2)$
- $x^2 - x + 1 = (x + \omega)(x + \omega^2)$
- $x^2 + xy + y^2 = (x - y\omega)(x - y\omega^2)$
- $x^2 - xy + y^2 = (x + \omega y)(x + y\omega^2)$
- $x^2 + y^2 = (x + iy)(x - iy)$
- $x^3 + y^3 = (x + y)(x + y\omega)(x + y\omega^2)$
- $x^3 - y^3 = (x - y)(x - y\omega)(x - y\omega^2)$
- $x^2 + y^2 + z^2 - xy - yz - zx = (x + y\omega + y\omega^2)(x + y\omega^2 + z\omega)$ or $(x\omega + y\omega^2 + z)(x\omega^2 + y\omega + z)$ or $(x\omega + y + z\omega^2)(x\omega^2 + y + z\omega)$
- $x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x + \omega y + \omega^2 z)(x + \omega^2 y + \omega z)$

SPEED BOOSTER

$$1 + \omega^n + \omega^{2n} = \begin{cases} 3, \text{ when } n \text{ is a multiple of } 3 \\ 0, \text{ when } n \text{ is not a multiple of } 3 \end{cases}$$

De-Moivre's Theorem

(i) If n is an integer, then

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

(ii) If n is a rational number, then $(\cos n\theta + i\sin n\theta)$ is one of the value of $(\cos\theta + i\sin\theta)^n$

Note If $z = (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)(\cos\theta_3 + i\sin\theta_3) \ldots (\cos\theta_n + i\sin\theta_n)$, then $z = \cos(\theta_1 + \theta_2 + \theta_3 + \ldots + \theta_n) + i\sin(\theta_1 + \theta_2 + \theta_3 + \ldots + \theta_n)$

nth Roots of Any Complex Number

If $z = r(\cos\theta + i\sin\theta)$ and n is a positive integer, then

$$z^{1/n} = r^{1/n}\left(\cos\left(\frac{2k\pi + \theta}{n}\right) + i\sin\left(\frac{2k\pi + \theta}{n}\right)\right)$$

where, $k = 0, 1, 2, 3, \ldots, (n - 1)$

nth Roots of Unity

The equation $x^n = 1$ has n roots which are called the nth roots of unity.

$\therefore$ $x^n = 1 \Rightarrow x^n = \cos 2k\pi + i\sin 2k\pi$

$x = (\cos 2k\pi + i\sin 2k\pi)^{1/n}$

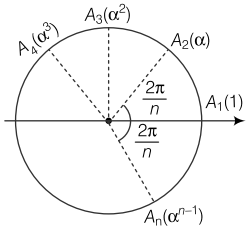
$x = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}$

where, $k = 0, 1, 2, \ldots, n-1$. Let $\alpha = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$, then the nth roots of unity are $1, \alpha, \alpha^2, \alpha^3, \ldots, \alpha^{n-1}$ which are in GP.

SPEED BOOSTER

- $1+\alpha+\alpha^2+\alpha^3+\ldots+\alpha^{n-1}=0$
- $1\times\alpha\times\alpha^2\times\alpha^3\times\ldots\times\alpha^{n-1}=\begin{cases}-1, & n \text{ is even}\\ 1, & n \text{ is odd}\end{cases}$

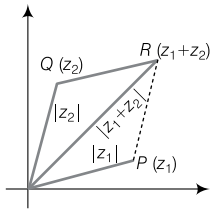
The points represented by the nth roots of unity are located at the vertices of a regular polygon of n sides inscribed in a unit circle having centre at the origin, one vertex on the positive real axis.



Use of Complex Numbers in Coordinate Geometry

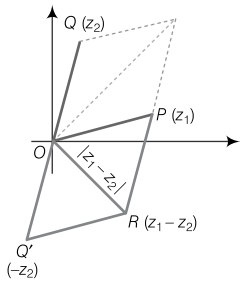
Addition of Two Complex Numbers

Sum of complex numbers can be interpreted in terms of geometry as a vector. Sum of two complex numbers z_1 and z_2 is a diagonal of parallelogram formed by the vectors corresponds to complex numbers z_1 and z_2.



Difference of Complex Numbers

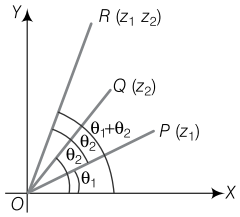
Difference of two complex numbers z_1 and z_2 can be considered as sum of the complex numbers z_1 and $-z_2$.



Multiplication of Complex Numbers

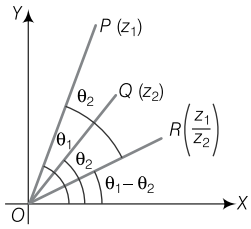
When two complex numbers $z_1 = r_1e^{i\theta_1}$ and $z_2 = r_2e^{i\theta_2}$ are represented by the vectors **OP** and **OQ**, respectively. Then, product z_1 with z_2 rotate z_1 anti-clockwise by an angle θ_2 $(\theta_2 > 0)$ or clockwise when $\theta_2 < 0$. The modulus of the complex number z_1z_2 is $|z_1||z_2|$.

$\therefore \quad z_1z_2 = r_1r_2e^{i(\theta_1+\theta_2)}$



Division of Two Complex Numbers

When two complex numbers $z_1 = r_1e^{i\theta_1}$ and $z_2 = r_2e^{i\theta_2}$ are represented by the vectors **OP** and **OQ**, respectively. Then, division of z_1 by z_2 rotate the vector z_1 in clockwise by an angle θ_2 and modulus of the complex number is $\left|\frac{z_1}{z_2}\right|$.

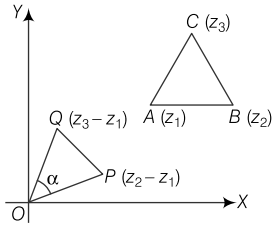


$\therefore \quad \frac{z_1}{z_2} = \frac{r_1}{r_2}e^{i(\theta_1-\theta_2)}$

By the concept of multiplication or division an important concept of rotation is generated which helps the students to find many problem in coordinate geometry.

Coni's Theorem (Rotation Theorem)

If z_1, z_2 and z_3 are the vertices of ΔABC described in counter clockwise sense, then



$$\mathbf{OQ} = \mathbf{OP}^{i\alpha}$$

$$\frac{\mathbf{OQ}}{|\mathbf{OQ}|} = \frac{\mathbf{OP}}{|\mathbf{OP}|}e^{i\alpha}$$

$$\frac{z_3 - z_1}{|z_3 - z_1|} = \frac{z_2 - z_1}{|z_2 - z_1|} e^{i\alpha}$$

$$\frac{z_3 - z_1}{z_2 - z_1} = \frac{|z_3 - z_1|}{|z_2 - z_1|} e^{i\alpha}$$

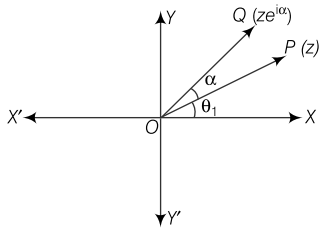
If $|z_3 - z_1| = |z_2 - z_1|$

Then, $z_3 - z_1 = (z_2 - z_1)\, e^{i\alpha}$

i.e. if we have to find the new position of complex number when z_1 rotated by α angle, then new complex number is $z_2 = z_1\, e^{i\alpha}$.

POINTS TO REMEMBER

(i)



(a) $ze^{i\alpha}$ is the complex number whose modulus is r and argument $\theta + \alpha$.

(b) Multiplication by $e^{-i\alpha}$ to z rotates the vector **OP** in clockwise sense through an angle α.

(ii) If z_1, z_2, z_3 and z_4 are the affixes of the points A, B, C and D, respectively in the argand plane.

(a) AB is inclined to CD at the angle $\arg\left(\frac{z_2 - z_1}{z_4 - z_3}\right)$.

(b) If CD is inclined at 90° to AB, then $\arg\left(\frac{z_2 - z_1}{z_4 - z_3}\right) = \pm\frac{\pi}{2}$.

(c) If z_1 and z_2 are fixed complex numbers, then the locus of a point z satisfying $\arg\left(\frac{z - z_1}{z - z_2}\right) = \pm\frac{\pi}{2}$.

Triangle Inequalities

It is an important concept to find maximum and minimum values. If z_1 and z_2 are two complex numbers, then

(i) $|z_1 + z_2| \le |z_1| + |z_2|$, i.e. sum of the two sides is always greater than third side.

(ii) $|z_1 + z_2| \ge |z_1| - |z_2|$, i.e. difference between two sides of triangle is always less than third side.

(iii) $||z_1| - |z_2|| \le |z_1 + z_2| \le |z_1| + |z_2|$

IMPORTANT POINTS

- Distance between $A(z_1)$ and $B(z_2)$ is given by $AB = |z_2 - z_1|$
- The points $P(z)$ divides the line segment joining $A(z_1)$ and $B(z_2)$ in the ratio $m : n$.
  (a) For internal division, $z = \frac{mz_2 + nz_1}{m + n}$
  (b) For external division, $z = \frac{mz_2 - nz_1}{m - n}$
- Let ABC be a triangle with vertices $A(z_1)$, $B(z_2)$ and $C(z_3)$, then
  (a) Centroid $G(z)$ of the triangle is given by $z = \frac{1}{3}(z_1 + z_2 + z_3)$
  (b) Incentre of $I(z)$ of the triangle is given by $z = \frac{az_1 + bz_2 + cz_3}{a + b + c}$, where a, b and c are sides of triangle.
- Area of ΔABC with vertices $A(z_1)$, $B(z_2)$, $C(z_3)$ is given by $\Delta = \frac{1}{2}\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix}$.

For an equilateral triangle $z_1^2 + z_2^2 + z_3^2 = z_2z_3 + z_3z_1 + z_1z_2$

- The general equation of straight line is $\bar{a}z + a\bar{z} + b = 0$, where a is complex number, b is real number.
- If z_1 and z_2 are two fixed points, then $|z - z_1| = |z - z_2|$ represents perpendicular bisector of the line segment joining $A(z_1)$ and $B(z_2)$.
- The equation of a circle with centre z_0 and radius r is $|z - z_0| = r$ or $z\bar{z} - \bar{z}_0 z - z_0\bar{z} + z_0\bar{z}_0 - r^2 = 0$.
- General equation of circle is $z\bar{z} + a\bar{z} + \bar{a}z + b = 0$, where a is complex number, b is real number.
- If z_1 and z_2 are two fixed points and $k > 0$, $k \ne 1$ is a real number, then $\frac{|z - z_1|}{|z - z_2|} = k$ represents a circle. For $k = 1$, it represent perpendicular bisector of the segment joining $A(z_1)$, $B(z_2)$.
- The diameter form of circle is $(z - z_1)(\bar{z} - \bar{z}_2) + (z - z_2)(\bar{z} - \bar{z}_1) = 0$ where $A(z_1)$ and $B(z_2)$ are end points of diameter.

MATHEMATICS IN action!

The First Calculator - Abacus

An **abacus** (plurals **abacuses** or **abaci**), also called a **counting frame**, is a calculating tool for performing arithmetical processes, often constructed as a wooden frame with beads sliding on wires. The user, called an **abacist**, slides counters by hand on rods or in grooves. It was in use centuries before the adoption of the written Hindu-Arabic numeral system and is still widely used by merchants and clerks in China, Japan (/entry/Japan), Africa (/entry/Africa) and elsewhere.

The abacus is a simple, inexpensive, yet powerful tool for calculation. Although electronic calculators are replacing it for practical reasons, the use of the abacus continues to be taught in technologically advanced countries such as Japan (/entry/Japan). Just as with popular board games such as chess, shogi (/entry/Shogi), and go (/entry/Go), local, regional and national competitions in abacus use are held for students.

The first abacus was most likely based on a flat stone covered with sand or dust. Words and letters were drawn in the sand; eventually numbers were added and pebbles used to aid calculations. The Babylonians used this dust abacus as early as 2400 B.C.E. The origin of the counter abacus with strings is obscure, but India (/entry/India), Mesopotamia or Egypt (/entry/Egypt) are seen as probable points of origin. China played an essential part in the development and evolution of the abacus.

MASTER STROKES

1. The square root of $x+\sqrt{-x^4-x^2-1}$ is

(a) $(x^2+x+1)+i(x^2-x+1)$

(b) $\pm\frac{1}{\sqrt{2}}\{\sqrt{x^2+x+1}+i\sqrt{x^2-x+1}\}$

(c) $\frac{1}{\sqrt{2}}(\sqrt{x^2-1}+i\sqrt{x^2+1})$

(d) None of the above

2. The value of $(1+i)^4$ is

(a) $4i$ (b) -4 (c) $-4i$ (d) 4

3. The conjugate of $\frac{2+3i}{1+2i}$ is

(a) $\frac{8+i}{5}$ (b) $\frac{8-i}{5}$ (c) $\frac{2+3i}{5}$ (d) $\frac{2-3i}{5}$

4. $(1-i)(1+2i)(1-3i)$ is equal to

(a) $1-6i$ (b) $8-6i$ (c) $6-8i$ (d) $6+8i$

5. Which of the following is correct?

(a) $2+3i>3+2i$ (b) $2-3i>3+2i$

(c) $3+4i>2+3i$ (d) None of these

6. The complex numbers z_1, z_2 and z_3 satisfying $\frac{z_1-z_3}{z_2-z_3}=\frac{1-i\sqrt{3}}{2}$ are vertices of a triangle which is

(a) of zero area (b) right angle isosceles

(c) equilateral (d) obtuse angled isosceles

7. If $x+iy=\frac{a+i}{a-i}$, $x, y, a\in R$, then $ay-x$ is equal to

(a) a (b) x (c) y (d) 1

8. If the imaginary part of $\frac{2z+1}{iz+1}$ is $-z$, then the locus of the point represented by z is a

(a) straight line (b) parabola

(c) circle (d) None of these

9. Let $A=\{z:\text{Im}(z)\geq 1\}$, $B=\{z:|z-2-i|=3\}$ and $C=\{z:\text{Re}((1-i)z)=\sqrt{2}\}$ be three sets of complex numbers. Then, the number of elements in the set $A\cap B\cap C$ is

(a) 0 (b) 1 (c) 2 (d) ∞

10. If ω is a complex root of the equation $z^3=1$, then $\omega+\omega^{\left(\frac{1}{2}+\frac{3}{8}+\frac{9}{32}+\frac{27}{128}+\ldots\right)}$ is

(a) 0 (b) 1 (c) -2 (d) -1

11. If z_1, z_2, z_3 are complex numbers such that $|z_1|=|z_2|=|z_3|=\left|\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}\right|=1$, then $|z_1+z_2+z_3|$ is

(a) equal to 1 (b) less than 1

(c) greater than 1 (d) equal to 3

12. The amplitude of $\sin\frac{\pi}{5}+i\left(1-\cos\frac{\pi}{5}\right)$ is

(a) $\frac{2\pi}{5}$ (b) $\frac{\pi}{5}$ (c) $\frac{\pi}{15}$ (d) $\frac{\pi}{10}$

13. If $\sin 6\theta=32\cos^5\theta\sin\theta-32\cos^3\theta\sin\theta+3x$, then x is equal to

(a) $\cos\theta$ (b) $\cos 2\theta$ (c) $\sin\theta$ (d) $\sin 2\theta$

14. If $|z+4|\leq 3$, then greatest and least values of $|z+1|$ are

(a) 4, 1 (b) 6, 0 (c) 6, 1 (d) 4, 0

15. If $x=a+b+c$, $y=a\alpha+b\beta+c$ and $z=a\beta+b\alpha+c$, where α and β are complex cube roots of unity, then xyz is

(a) $2(a^3+b^3+c^3)$ (b) $2(a^3-b^3-c^3)$

(c) $a^3+b^3+c^3-3abc$ (d) $a^3-b^3-c^3$

16. If z_r, $r=1, 2, 3, \ldots, 50$ are the roots of the equation $\sum_{r=0}^{50}(z)^r=0$, then the value of $\sum_{r=1}^{50}\frac{1}{z_r-1}$ is

(a) -85 (b) 25 (c) 75 (d) -25

17. Let ω be the solution of $x^3-1=0$ with $\text{Im}(\omega)>0$. If $a=2$ with b and c satisfying

$$[a\,b\,c]\begin{bmatrix}1&9&7\\2&8&7\\7&3&7\end{bmatrix}=[0\;0\;0]$$

then the value of $\frac{3}{\omega^a}+\frac{1}{\omega^b}+\frac{3}{\omega^c}$ is equal to

(a) -2 (b) 2 (c) 3 (d) -3

18. If $z=x+iy$ and $\omega=\frac{1-zi}{z-i}$, then $|\omega|=1$ implies that, in the complex plane

(a) z lies on the imaginary axis

(b) z lies on the real axis

(c) z lies on the unit circle

(d) None of the above

19. The complex number z satisfying $z+|z|=1+7i$, then the value of $|z|^2$ equals

(a) 625 (b) 169 (c) 49 (d) 25

20. Given that the equation $z^2+(p+iq)z+(r+is)=0$, where p, q, r and s are real and non-zero has a real root, then

(a) $pqr=r^2+p^2s$ (b) $prs=q^2+r^2p$

(c) $qrs=p^2+s^2q$ (d) $pqs=s^2+q^2r$

21. Let $z_r=(1\leq r\leq 4)$ be complex numbers such that $|z_r|=\sqrt{r+1}$ and $|30z_1+20z_2+15z_3+12z_4|=K|z_1z_2z_3+z_2z_3z_4+z_3z_4z_1+z_4z_1z_2|$, then the value of K is equal to

(a) $\sqrt{20}$ (b) $\sqrt{24}$ (c) $\sqrt{30}$ (d) $\sqrt{10}$

22. If $|z-3| = \min\{|z-1|, |z-5|\}$, then $\mathrm{Re}(z)$ equals

(a) 2 (b) $\frac{5}{2}$

(c) $\frac{7}{2}$ (d) 6

23. If α is the nth root of unity, then $1 + 2\alpha + 3\alpha^2 + \ldots$ to n terms equal to

(a) $\frac{-n}{(1-\alpha)^2}$ (b) $\frac{-n}{1-\alpha}$

(c) $\frac{-2n}{1-\alpha}$ (d) $\frac{-2n}{(1-\alpha)^2}$

24. Which of the following represents a point in an argand plane, equidistant from the roots of the equation $(z+1)^4 = 16z^4$?

(a) (0, 0) (b) $\left(\frac{-1}{3}, 0\right)$

(c) $\left(\frac{1}{3}, 0\right)$ (d) $\left(0, \frac{2}{\sqrt{5}}\right)$

25. If $|z| < \sqrt{2} - 1$, then $|z^2 + 2z\cos\alpha|$ is

(a) less than 1 (b) $\sqrt{2} + 1$

(c) $\sqrt{2} - 1$ (d) None of these

26. If $8iz^3 + 12z^2 - 18z + 27i = 0$, then

(a) $|z| = \frac{3}{2}$ (b) $|z| = \frac{2}{3}$ (c) $|z| = 1$ (d) $|z| = \frac{3}{4}$

27. z_1 and z_2 are the roots of $3z^2 + 3z + b = 0$, if $O(0)$, (z_1), (z_2) form an equilateral triangle, then b equals

(a) 1 (b) 2 (c) 3 (d) 4

28. If z and w are two complex numbers simultaneously satisfying the equations $z^3 + w^5 = 0$ and $z^2w^{-4} = 1$, then

(a) z and w both are purely real

(b) z is purely real and w is purely imaginary

(c) w is purely real and z is purely imaginary

(d) z and w both are imaginary

29. If $\alpha = e^{i(2\pi/n)}$, then $(11-\alpha)(11-\alpha^2)\ldots(11-\alpha^{n-1})$ is equal to

(a) 11^{n-1} (b) $\frac{11^{n-1}-1}{11}$ (c) $\frac{11^{n-1}-1}{10}$ (d) $\frac{11^n-1}{10}$

30. The length of perpendicular from $p(z_0)$ to the line $a\bar{z} + \bar{a}z + b = 0$ is

(a) $\frac{|a\bar{z}_0 + \bar{a}z_0 + b|}{|a|}$ (b) $\frac{|a\bar{z}_0 + \bar{a}z_0 + b|}{2|a|}$

(c) $\frac{|az_0 + \bar{a}\bar{z}_0 + b|}{|a|}$ (d) $\frac{|az_0 + \bar{a}\bar{z}_0 + b|}{2|a|}$

31. A man walks a distance of 3 units from the origin towards the North-East ($N\,45^\circ E$) direction. From there, he walks a distance of 4 units towards the North-West ($N\,45^\circ W$) direction to reach a point P. Then, the position of P in the argand plane is

(a) $3e^{i\pi/4} + 4i$ (b) $(3-4i)\,e^{i\pi/4}$

(c) $(4+3i)\,e^{i\pi/4}$ (d) $(3+4i)\,e^{i\pi/4}$

32. If $\omega = e^{i\frac{2\pi}{3}}$ and a, b, c, x, y, z be non-zero complex numbers such that

$a + b + c = x$, $a + b\omega + c\omega^2 = y$, $a + b\omega^2 + c\omega = z$

Then, the value of $\frac{|x|^2 + |y|^2 + |z|^2}{|a|^2 + |b|^2 + |c|^2}$ is

(a) 3 (b) 6 (c) 9 (d) 1

33. If $x = 2 + 5i$ and $2\left[\frac{1}{1!9!} + \frac{1}{3!7!}\right] + \frac{1}{5!5!} = \frac{2^a}{b!}$, then the value of $(x^3 - 5x^2 + 33x - 19)$ is equal to

(a) a (b) b (c) $a - b$ (d) $a + b$

34. If z_1 and $\bar{z}_1$ represent adjacent vertices of a regular polygon of n sides whose centre is origin and if $\frac{\mathrm{Im}(z_1)}{\mathrm{Re}(z_1)} = \sqrt{2} - 1$, then n is equal to

(a) 8 (b) 16 (c) 24 (d) 32

35. $\sin^{-1}\left\{\frac{1}{i}(z-1)\right\}$, where z is non-real and $i = \sqrt{-1}$, can be the angle of triangle, if

(a) $\mathrm{Re}(z) = 1, \mathrm{Im}(z) = 2$ (b) $\mathrm{Re}(z) = 1, 0 \le \mathrm{Im}(z) \le 1$

(c) $\mathrm{Re}(z) + \mathrm{Im}(z) = 0$ (d) None of these

36. If $\log_{\sqrt{3}}\left(\frac{|z|^2 - |z| + 1}{2 + |z|}\right) > 2$, then the locus of z is

(a) $|z| = 5$ (b) $|z| < 5$

(c) $|z| > 5$ (d) None of these

37. Reflection of the line $(2+3i)z + (2-3i)\bar{z} = 0$ on the real axis is

(a) $(2-3i)z - (2+3i)\bar{z} = 0$

(b) $(2+3i)\bar{z} - (2-3i)z = 0$

(c) $(2+3i)z + (2-3i)\bar{z} = 0$

(d) $(2-3i)z + (2+3i)\bar{z} = 0$

38. Let complex numbers α and $1/\bar{\alpha}$ lies on circles $(x-x_0)^2 + (y-y_0)^2 = r^2$ and $(x-x_0)^2 + (y-y_0)^2 = 4r^2$, respectively. If $z_0 = x_0 + iy_0$ satisfies the equation $2|z_0|^2 = r^2 + 2$, then $|\alpha|$ is equal to

(a) $\frac{1}{\sqrt{2}}$ (b) $\frac{1}{2}$ (c) $\frac{1}{\sqrt{7}}$ (d) $\frac{1}{3}$

39. If $\omega = \frac{z}{z - i/3}$ and $|\omega| = 1$, then z lies on

(a) parabola (b) a straight line

(c) a circle (d) an ellipse

40. If the roots of the equation $z^4 + az^3 + (12+9i)z^2 + bz = 0$ are the vertices of a square, then $|a|^2 + |b|^2$ is equal to

(a) 270 (b) 280 (c) 290 (d) None of these

41. If $|z| = \max\{|z-1|, |z+1|\}$, then

(a) $|z + \bar{z}| = \frac{1}{2}$ (b) $z + \bar{z} = 1$

(c) $|z + \bar{z}| = 1$ (d) $z \in \phi$

42. If $x^2 + x + 1 = 0$, then the numerical value of $\left(x+\frac{1}{x}\right)^2+\left(x^2+\frac{1}{x^2}\right)^2+\left(x^3+\frac{1}{x^3}\right)^2+\ldots+\left(x^{27}+\frac{1}{x^{27}}\right)^2$ is

(a) 54 (b) 36 (c) 27 (d) 18

43. The expression $\left(\frac{1+i\tan\alpha}{1-i\tan\alpha}\right)^n-\left(\frac{1+i\tan n\alpha}{1-i\tan n\alpha}\right)$, when simplified, reduces to

(a) 0 (b) $2\sin n\alpha$
(c) $2\cos n\alpha$ (d) None of these

44. Intercept made by the circle $z\bar{z}+\bar{\alpha}z+\alpha\bar{z}+r=0$ on the real axis on the complex plane is

(a) $\sqrt{(\alpha+\bar{\alpha})-r}$ (b) $\sqrt{(\alpha+\bar{\alpha})^2-2r}$
(c) $\sqrt{(\alpha+\bar{\alpha})^2+r}$ (d) $\sqrt{(\alpha+\bar{\alpha})^2-4r}$

45. If $z_1 = 2 + 3i$, $z_2 = 3 - 2i$ and $z_3 = -1 - 2\sqrt{3}i$, then which of the following is true?

(a) $\arg\left(\frac{z_3}{z_2}\right)=\arg\left(\frac{z_3-z_1}{z_2-z_1}\right)$

(b) $\arg\left(\frac{z_3}{z_2}\right)=\arg\left(\frac{z_2}{z_1}\right)$

(c) $\arg\left(\frac{z_3}{z_2}\right)=2\arg\left(\frac{z_3-z_1}{z_2-z_1}\right)$

(d) $\arg\left(\frac{z_3}{z_2}\right)=\frac{1}{2}\arg\left(\frac{z_3-z_1}{z_2-z_1}\right)$

46. Let α and β be real and z be complex number, if $z^2+\alpha z+\beta=0$ has two distinct roots on the line $\mathrm{Re}(z)=1$, then it is necessary that

(a) $\beta\in(-1,0)$ (b) $|\beta|=1$ (c) $\beta\in(1,\infty)$ (d) $\beta\in(0,1)$

47. If $|z-1|=1$ and $\arg z=\theta(z\neq 0)$ and $0<\theta<\frac{\pi}{2}$, then $1-\frac{2}{z}$ is equal to

(a) $\tan\theta$ (b) $i\tan\theta$ (c) $\tan\theta/2$ (d) $i\tan\theta/2$

48. Number of complex numbers z such that $|z|=1$ and $\left|\frac{z}{\bar{z}}+\frac{\bar{z}}{z}\right|=1$ is

(a) 4 (b) 6 (c) 8 (d) More than 8

49. Let $A(z_1)$, $B(z_2)$ and $C(z_3)$ be the vertices of ΔABC such that $z_3+i\omega z_2=(1+i\omega)z_1$, where $\omega(\neq 1)$ be a cube root of unity, then $\arg\left(\frac{z_3-z_1}{z_2+z_3-2z_1}\right)$ is equal to

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$
(c) $\frac{\pi}{12}$ (d) $\frac{\pi}{4}$

50. If α is non-real and $\alpha=\sqrt[5]{1}$, then the value of $2^{|1+\alpha+\alpha^2+\alpha^{-2}-\alpha^{-1}|}$ is equal to

(a) 4 (b) 2 (c) 1 (d) None

51. Let $\omega(\neq 1)$ be a complex cube root of unity. If $(3-3\omega+2\omega^2)^{4n+3}+(2+3\omega-3\omega^2)^{4n+3}+(-3+2\omega+2\omega^2)^{4n+3}=0$, then the set of possible value(s) on n is(are)

(a) N (b) $\{3k:k\in N\}$
(c) $N-\{3k:k\in N\}$ (d) $\{6k:k\in N\}$

52. For any integer k, let $a_k=\cos\left(\frac{k\pi}{7}\right)+i\sin\left(\frac{k\pi}{7}\right)$, where $i=\sqrt{-1}$. The value of the expression $\frac{\sum_{k=1}^{12}|\alpha_{k+1}-\alpha_k|}{\sum_{k=1}^{3}|\alpha_{4k-1}-\alpha_{4k-2}|}$ is

(a) 8 (b) 6 (c) 4 (d) 2

53. If $f(n)=\cot^2\frac{\pi}{n}+\cot^2\frac{2\pi}{n}+\cot^2\frac{3\pi}{n}+\ldots+\frac{\cot^2(n-1)\pi}{n}$, $n>1$ and $n\in N$. Then, $\lim_{n\to\infty}\frac{f(n)}{n^2}$ is equal to

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) $\frac{2}{3}$ (d) 1

54. If z is a complex number satisfying $|z|^2+2(z+\bar{z})+3i(z-\bar{z})+4=0$, then complex number $z+3+2i$ lies on

(a) circle with centre $1-5i$ and radius 4
(b) circle with centre $1+5i$ and radius 4
(c) circle with centre $1+5i$ and radius 3
(d) circle with centre $1-5i$ and radius 3

55. The maximum area of the triangle formed by the complex coordinates z, z_1, z_2 which satisfy the relations

$|z-z_1|=|z-z_2|$ and $\left|z-\frac{z_1+z_2}{2}\right|\le r$, where $r>|z_1-z_2|$ is

(a) $\frac{1}{2}|z_1-z_2|^2$ (b) $\frac{1}{2}|z_1-z_2|r$
(c) $\frac{1}{2}|z_1-z_2|^2r^2$ (d) $\frac{1}{2}|z_1-z_2|r^2$

Answers

1. (b)	2. (b)	3. (a)	4. (c)	5. (d)	6. (c)	7. (d)	8. (a)	9. (b)	10. (d)
11. (a)	12. (d)	13. (d)	14. (b)	15. (c)	16. (d)	17. (a)	18. (b)	19. (a)	20. (d)
21. (c)	22. (a)	23. (b)	24. (c)	25. (a)	26. (a)	27. (a)	28. (a)	29. (d)	30. (b)
31. (d)	32. (a)	33. (b)	34. (b)	35. (b)	36. (c)	37. (d)	38. (c)	39. (b)	40. (c)
41. (d)	42. (b)	43. (a)	44. (d)	45. (a)	46. (c)	47. (b)	48. (c)	49. (c)	50. (a)
51. (c)	52. (c)	53. (b)	54. (c)	55. (b)					

TARGET JEE 2017

TEST RIDER

Comprehensive Simulator Test Series for JEE Main & Advanced

JEE MAIN

(PAPER ONE PREP UP)

HALF TEST 2

(XII SYLLABUS) FULLY SOLVED

Mock Questions from Class XIIth Syllabus with Complete Solutions

INSTRUCTIONS

- This test consists of 30 questions and each question is allotted 4 marks for correct response.
- Candidates will be awarded marks as stated above for correct response of each question. 1/4 mark will be deducted for indicating incorrect response of each question. No deduction from the total score will be made if no response is indicated for an item in the answer sheet.
- There is only one correct response for each question. Filling up more than one response in any question will be treated as wrong response and marks for wrong response will be deducted according as per instructions.

1. If $D_k = \begin{vmatrix} 1 & n & 2 \\ 2k & n^2+n+2 & n+3 \\ 2k-1 & n^2 & 2n^2+5n+1 \end{vmatrix}$, then $\sum_{k=1}^{n} D_k$ is divisible by

(a) $n+3$ (b) $3n+2$
(c) $2n+1$ (d) $n-1$

2. The minimum value of $f(x) = \dfrac{x-[x]-2}{3-x+[x]}$ is (where $[\cdot]$ is the greatest integer function)

(a) $\frac{1}{3}$ (b) $\frac{1}{2}$ (c) 0 (d) $-\frac{2}{3}$

3. If $f\left(x+\frac{1}{2}\right)+f\left(x-\frac{1}{2}\right)=f(x)$, $\forall\, x \in R$, then $f(x-3)+f(x+3)$ is equal to

(a) $f(x)$ (b) $2f(x)$ (c) 0 (d) $6f(x)$

4. If $(\sin^{-1}x)^2+\left[\tan^{-1}\left\{\left(1+\frac{1}{x}\right)\sqrt{\frac{1-x}{1+x}}\right\}\right]^2=\frac{5\pi^2}{8}$, then x is equal to

(a) $-\frac{1}{2}$ (b) ± 1 (c) $-\frac{1}{\sqrt{2}}$ (d) $\pm\frac{\sqrt{3}}{2}$

5. If $0<x<1$, then $\sqrt{1+x^2}[\{\sin(\cot^{-1}x)+\sin(\tan^{-1}x)\}^2-1]$ is equal to

(a) $\dfrac{2x}{\sqrt{1+x^2}}$ (b) $2x$
(c) $2x\sqrt{1+x^2}$ (d) $\sqrt{1+x^2}$

6. If A is idempotent matrix and I is identity matrix such that $(I+A)^n = I+(2^n+k)A$. Then, the value of k is

(a) 0 (b) 1
(c) -1 (d) None of these

7. If $f(x-y)$, $f(x)f(y)$ and $f(x+y)$ are in AP, $\forall\, x, y \in R$ and $f(0)=1$, then

(a) $f'(-2)=f'(2)$
(b) $f'(-3)=-f'(3)$
(c) $f'(-2)+f'(2)=0$
(d) None of the above

8. Let the sequence $< b_n >$ of real numbers satisfy the recurrence relation : $b_{n+1}=\frac{1}{3}\left(2b_n+\frac{125}{b_n^2}\right)$, $b_n \neq 0$, then $\lim\limits_{n\to\infty} b_n$ is equal to

(a) 0 (b) ∞ (c) 5 (d) $\frac{2}{3}$

9. If $af(x+1)+bf\left(\dfrac{1}{x+1}\right)=x$, $x\neq -1$, $a\neq b$, then $f'(1)$ is equal to

(a) $a+b$ (b) $\dfrac{1}{a+b}$
(c) a^2-b^2 (d) None of these

10. The length of the largest continuous interval in which the function $f(x)=4x-\tan 2x$ is monotonic, is

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$
(c) $\frac{\pi}{8}$ (d) $\frac{\pi}{16}$

11. If the function $f(x) = 2\tan x + (2a+1)\log_e|\sec x| + (a-2)x$ is increasing on R, then

(a) $a \in \left(\frac{1}{2}, \infty\right)$ (b) $a \in \left(-\frac{1}{2}, \frac{1}{2}\right)$

(c) $a \in \left(-\infty, -\frac{1}{2}\right)$ (d) None of these

12. If $0 < x < 2\pi$, $I_n = \int \left[\frac{1+\cos x}{2}\right]^{n/2} dx$, then $\sum_{n=0}^{\infty} (I_n)$ is equal to

(a) $-\cot\frac{x}{2} - \text{cosec}\frac{x}{2} - \frac{x}{2} + C$

(b) $-2\cot\frac{x}{2} - 2\text{cosec}\frac{x}{2} + C$

(c) $-\text{cosec}\, x + \cot x + x + C$

(d) None of the above

13. If $\int \tan^4 x\, dx = K\tan^3 x + L\tan x + f(x)$, then

(a) $K = \frac{1}{3}, L = -1, f(x) = x + C$

(b) $K = 1, L = -1, f(x) = -x + C$

(c) $K = -1, L = 1, f(x) = 2x + C$

(d) $K = \frac{1}{2}, L = \frac{1}{3}, f(x) = 3x + C$

14. Let $f:(0,\infty) \to R$ and $\int_0^{2x} (1-t) f(t)\, dt = -2x^3 - \frac{x^2}{2} + 2x + K$, where K is constant, then $\sum_{r=1}^{8} f(r)$ is equal to

(a) 43 (b) 57

(c) 28 (d) 35

15. The area bounded by the curve $xy - 3x - 2y - 10 = 0$, X-axis and straight lines $x = 3$, $x = 4$ is

(a) $16\log 2 - 13$ (b) $16\log 2 - 3$

(c) $16\log 2 + 3$ (d) None of these

16. If the curve satisfying the equation $(x^2 + y^2)\, dy = xy\, dx$ and passing through points $(1, 1)$ and $(k, \sqrt{e})$, then the value of k is

(a) $\sqrt{2e}$ (b) $\sqrt{e - \frac{1}{2}}$ (c) $\sqrt{\frac{e-1}{2}}$ (d) $\sqrt{\frac{e+1}{2}}$

17. The differential equation of the family of circles passing through the fixed points $(a, 0)$ and $(-a, 0)$ is

(a) $y_1(y^2 - x^2) + 2xy + a^2 = 0$

(b) $y_1 y^2 + xy + a^2 x^2 = 0$

(c) $y_1(y^2 - x^2 + a^2) + 2xy = 0$

(d) None of the above

18. The values of x for which the angle between the vectors $\mathbf{a} = x\hat{\mathbf{i}} - 3\hat{\mathbf{j}} - \hat{\mathbf{k}}$ and $\mathbf{b} = 2x\hat{\mathbf{i}} + x\hat{\mathbf{j}} - \hat{\mathbf{k}}$ is acute and the angle between the vectors $\mathbf{b}$ and the Y-axis lies between $\frac{\pi}{2}$ and π, are

(a) < 0 (b) > 0 (c) $-2, -3$ (d) $1, 2$

19. If $\mathbf{a}$ and $\mathbf{b}$ are two unit vectors inclined at an angle $\frac{\pi}{3}$, then the value of $(2\mathbf{a} + \mathbf{b}) \cdot [(\mathbf{a} \times \mathbf{b}) \times (\mathbf{a} - 2\mathbf{b})]$ is

(a) $-\frac{3}{5}$ (b) $\frac{15}{4}$

(c) $-\frac{7}{3}$ (d) $\frac{9}{2}$

20. The intercepts made on the X, Y and Z-axes by the plane which bisects the line joining the points $(1, 2, 3)$ and $(-3, 4, 5)$ at right angles are a, b and c respectively, then $2a + b + c$ is equal to

(a) $\frac{45}{2}$ (b) $\frac{27}{2}$ (c) 18 (d) 9

21. The equation of the plane through the intersection of the planes $x + y + z = 1$ and $2x + 3y - z + 4 = 0$ and parallel to X-axis is

(a) $y - 3z + 6 = 0$ (b) $3y - z + 6 = 0$

(c) $y + 3z + 6 = 0$ (d) $3y - 2z + 6 = 0$

22. A speaks truth in 60% cases and B speaks truth in 70% cases. The probability that they will not say the same thing while describing a single event is

(a) 0.54 (b) 0.46 (c) 0.38 (d) 0.94

23. Let A and B be two events such that $P(\overline{A \cup B}) = \frac{1}{6}$, $P(A \cap B) = \frac{1}{4}$ and $P(\overline{A}) = \frac{1}{4}$, where $\overline{A}$ stands for complement of event A. Then, A and B are

(a) mutually exclusive and independent

(b) independent but not equally likely

(c) equally likely but not independent

(d) equally likely and mutually exclusive

24. If $x + 4y = 14$ is a normal to the curve $y^2 = \alpha x^3 - \beta$ at $(2, 3)$, then the value of α, β are roots of the equation

(a) $x^2 - 9x + 14 = 0$ (b) $x^2 - 5x - 14 = 0$

(c) $x^2 + 5x - 14 = 0$ (d) $x^2 + 9x + 14 = 0$

25. If $f(x) = \begin{cases} |1 - 4x^2|, & 0 \le x < 1 \\ [x^2 - 2x], & 1 \le x < 2 \end{cases}$, where $[\cdot]$ denotes the greatest integer function, then $f(x)$ is

(a) differentiable for all x

(b) continuous at $x = 1$

(c) $f(x)$ is non-differentiable at $x = 1$

(d) None of the above

26. Let S be the set of all real numbers. Then, the relation $R = \{(a, b) : 1 + ab > 0\}$ on S is

(a) reflexive and symmetric but not transitive

(b) reflexive and transitive but not symmetric

(c) symmetric and transitive but not reflexive

(d) None of the above

27. The value of the integral $\int_a^{a+\pi} \frac{|\sin x + \cos x|}{|\sin x| + |\cos x|} dx$ is

(a) $a\pi$ (b) $2a\pi$

(c) $\frac{a\pi}{2}$ (d) independent of a

28. If $\vec{\alpha}+\vec{\beta}+\vec{\gamma}=a\vec{\delta}$ and $\vec{\beta}+\vec{\gamma}+\vec{\delta}=b\vec{\alpha}$, a and δ are non-collinear, then $\vec{\alpha}+\vec{\beta}+\vec{\gamma}+\vec{\delta}$ equals

(a) $a\vec{\alpha}$ *(b)* $b\vec{\delta}$ *(c)* 0 *(d)* $(a+b)\vec{\gamma}$

29. A line makes an angle θ with each of the X-axis and Z-axis. If the angle β, which it makes with Y-axis is such that $\sin^2\beta = 2\sin^2\theta$, then its directions cosines are

(a) $\pm\frac{1}{\sqrt{2}}, 0, \pm\frac{1}{\sqrt{2}}$ *(b)* $\frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2}$

(c) $\frac{1}{2\sqrt{2}}, \frac{\sqrt{3}}{2}, \frac{1}{2\sqrt{2}}$ *(d)* None of these

30. Consider the system of equations
$x - ay + az = 0,\ bx + y - bz = 0,$
$cx - cy - z = 0$
where a, b and c are non-zero real numbers.
Statement I If x, y, z are not all zero, then
$$1 + ab + bc + ca = 0$$
Statement II If $a + b + c > 1$, then $a^2 + b^2 + c^2 > 3$

(a) Both Statements are true and Statement II is correct explanation of Statement I
(b) Both Statements are true but Statement II is not correct explanation of Statement I
(c) Statement I is false and Statement II is true
(d) Statement I is true and Statement II is false

Answers with Explanation

1. ***(c)*** $\sum_{k=1}^{n} D_k = \begin{vmatrix} \sum_{k=1}^{n}(1) & n & 2 \\ \sum_{k=1}^{n}(2k) & n^2+n+2 & n+3 \\ \sum_{k=1}^{n}(2k-1) & n^2 & 2n^2+5n+1 \end{vmatrix}$

$$= \begin{vmatrix} n & n & 2 \\ n^2+n & n^2+n+2 & n+3 \\ n^2 & n^2 & 2n^2+5n+1 \end{vmatrix}$$

$$= \begin{vmatrix} n & 0 & 2 \\ n^2+n & 2 & n+3 \\ n^2 & 0 & 2n^2+5n+1 \end{vmatrix}$$

[applying $C_2 \to C_2 - C_1$]

$= 2[n(2n^2+5n+1) - 2n^2]$
$= 2n(2n^2+3n+1)$
$= 2n(n+1)(2n+1)$

Thus, $\sum_{k=1}^{n} D_k$ is divisible by $(2n+1)$.

2. ***(d)*** $f(x) = \frac{x-[x]-2}{3-x+[x]} = \frac{\{x\}-2}{3-\{x\}}$

$$= \frac{1}{3-\{x\}} - 1$$

As, $0 \le \{x\} < 1$
$\Rightarrow 0 \ge -\{x\} > -1$
$\Rightarrow 3 \ge 3-\{x\} > 2$
$\Rightarrow \frac{1}{3} \le \frac{1}{3-\{x\}} < \frac{1}{2}$
$\Rightarrow \frac{-2}{3} \le f(x) < \frac{-1}{2}$

Hence, minimum value of $f(x)$ is $-\frac{2}{3}$.

3. ***(b)*** $f\left(x+\frac{1}{2}\right) + f\left(x-\frac{1}{2}\right) = f(x)$

$\Rightarrow f(x+1) + f(x) = f\left(x-\frac{1}{2}\right)$

$\Rightarrow f(x+1) + f\left(x-\frac{1}{2}\right) = 0$

$\Rightarrow f\left(x+\frac{3}{2}\right) = -f(x)$

$\Rightarrow f(x+3) = -f\left(x+\frac{3}{2}\right) = f(x)$

So, $f(x)$ is periodic with period 3.
$\Rightarrow f(x-3) = f(x)$
$\therefore f(x-3) + f(x+3) = f(x) + f(x) = 2f(x)$

4. ***(c)*** $(\sin^{-1}x)^2 + \left[\tan^{-1}\left\{\left(1+\frac{1}{x}\right)\sqrt{\frac{1-x}{1+x}}\right\}\right]^2 = \frac{5\pi^2}{8}$

$\Rightarrow (\sin^{-1}x)^2 + \left[\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)\right]^2 = \frac{5\pi^2}{8}$

$\Rightarrow (\sin^{-1}x)^2 + (\cos^{-1}x)^2 = \frac{5\pi^2}{8}$

$\Rightarrow (\sin^{-1}x)^2 + \left(\frac{\pi}{2} - \sin^{-1}x\right)^2 = \frac{5\pi^2}{8}$

$\Rightarrow 2(\sin^{-1}x)^2 - \pi\sin^{-1}x - \frac{3\pi^2}{8} = 0$

$\Rightarrow \sin^{-1}x = \frac{\pi \pm 2\pi}{4}$

$\Rightarrow \sin^{-1}x = \frac{-\pi}{4}$

$\Rightarrow x = \frac{-1}{\sqrt{2}}$

5. ***(a)*** $\sqrt{1+x^2}[\{\sin(\cot^{-1}x) + \sin(\tan^{-1}x)\}^2 - 1]$

$$= \sqrt{1+x^2}\left[\left\{\sin\left(\sin^{-1}\frac{1}{\sqrt{1+x^2}}\right) + \sin\left(\sin^{-1}\frac{x}{\sqrt{1+x^2}}\right)\right\}^2 - 1\right]$$

$$= \sqrt{1+x^2}\left[\left\{\frac{1}{\sqrt{1+x^2}} + \frac{x}{\sqrt{1+x^2}}\right\}^2 - 1\right]$$

$$= \sqrt{1+x^2}\left[\frac{(1+x)^2}{1+x^2} - 1\right]$$

$$= \sqrt{1+x^2}\left[\frac{1+x^2+2x-1-x^2}{1+x^2}\right] = \frac{2x}{\sqrt{1+x^2}}$$

6. *(c)* $\because A$ and I both are idempotent matrix.

$\therefore \quad A^2 = A, I^2 = I$

Now, $\quad A^3 = A^2A = AA = A^2 = A$

So, $\quad A^4 = A^5 = A^6 = \ldots = A^n = A$

and $\quad I = I^2 = I^3 = \ldots = I^n = I$

$\therefore \quad (I + A)^n = I^n + {}^nC_1 I^{n-1}A + {}^nC_2 I^{n-2}A^2 + \ldots + {}^nC_n A^n$

$= I + {}^nC_1 A + {}^nC_2 A + \ldots + {}^nC_n A$

$= I + ({}^nC_1 + {}^nC_2 + .. + {}^nC_n)A$

$= I + (2^n - 1)A$

$\therefore \quad (I + A)^n = I + (2^n + k)A$ [given]

$\Rightarrow \quad k = -1$

7. *(c)* $f(x - y), f(x)f(y)$ and $f(x + y)$ are in AP.

$\therefore \quad 2f(x)f(y) = f(x - y) + f(x + y)$

Replacing x by 0 and y by x, we get

$2f(0)f(x) = f(-x) + f(x)$

$\Rightarrow \quad f(-x) + f(x) = 2f(x)$ $[\because f(0) = 1]$

$\Rightarrow \quad f(-x) = f(x)$

$\Rightarrow \quad -f'(-x) = f'(x)$

$\Rightarrow \quad f'(-x) + f'(x) = 0$

$\Rightarrow \quad f'(-2) + f'(2) = 0$

8. *(c)* Let $\lim_{n\to\infty} b_n = b$, then

$$b_{n+1} = \frac{1}{3}\left(2b_n + \frac{125}{b_n^2}\right)$$

$$\Rightarrow \lim_{n\to\infty} b_{n+1} = \frac{1}{3}\left\{2\lim_{n\to\infty} b_n + \frac{125}{\lim_{n\to\infty} b_n^2}\right\}$$

$$\Rightarrow \quad b = \frac{1}{3}\left\{2b + \frac{125}{b^2}\right\}$$

$$\Rightarrow \quad b - \frac{2b}{3} = \frac{125}{3b^2} \Rightarrow \frac{b}{3} = \frac{125}{3b^2}$$

$$\Rightarrow \quad b^3 = 125 \Rightarrow b = 5$$

9. *(b)* $a\,f(x + 1) + bf\left(\frac{1}{x + 1}\right) = x$ ...(i)

Replacing $(x + 1)$ by $\frac{1}{x + 1}$, we get

$$a\,f\left(\frac{1}{x + 1}\right) + b\,f(x + 1) = \frac{1}{x + 1} - 1$$

$$\Rightarrow \quad a\,f\left(\frac{1}{x + 1}\right) + b\,f(x + 1) = -\frac{x}{x + 1} \quad \ldots(ii)$$

Multiplying Eq. (i) by a, Eq. (ii) by b and subtracting, we get

$$(a^2 - b^2)\,f(x + 1) = ax + \frac{bx}{x + 1}$$

$$\Rightarrow \quad f(x + 1) = \frac{1}{(a^2 - b^2)}\left[ax + \frac{bx}{x + 1}\right]$$

$$\Rightarrow \quad f'(x + 1) = \frac{1}{(a^2 - b^2)}\left[a - \frac{b}{(x + 1)^2}\right]$$

Putting $x = 0$, we get

$$f'(1) = \frac{1}{a^2 - b^2}[a - b] = \frac{1}{a + b}$$

10. *(b)* $f(x) = 4x - \tan 2x$

$\Rightarrow f'(x) = 4 - 2\sec^2 2x = 2(1 - \tan^2 2x)$

For the continuous domain $\left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$

$f'(x) \geq 0$ in $\left[-\frac{\pi}{8}, \frac{\pi}{8}\right]$

and $f'(x) \leq 0$ in $\left(-\frac{\pi}{4}, -\frac{\pi}{8}\right] \cup \left[\frac{\pi}{8}, \frac{\pi}{4}\right)$

So, the required largest continuous interval is $\left[-\frac{\pi}{8}, \frac{\pi}{8}\right]$.

Thus, total length $= \frac{\pi}{8} - \left(-\frac{\pi}{8}\right) = \frac{\pi}{4}$

11. *(d)* If $f(x) = 2\tan x + (2a + 1)\log_e|\sec x| + (a - 2)x$ is increasing on R, then

$$f'(x) \geq 0$$

$$\Rightarrow 2\sec^2 x + (2a + 1)\tan x + (a - 2) \geq 0$$

$$\Rightarrow \quad 2\tan^2 x + (2a + 1)\tan x + a \geq 0$$

$$\Rightarrow \quad (2a + 1)^2 - 8a \leq 0$$

$$\Rightarrow \quad (2a - 1)^2 \leq 0$$

$$\Rightarrow \quad a = \frac{1}{2}$$

12. *(b)* $I_n = \int\left(\frac{1 + \cos x}{2}\right)^{n/2} dx$

$$= \int\left[\cos^2\frac{x}{2}\right]^{n/2} dx = \int \cos^n\frac{x}{2}\,dx$$

$$\therefore \sum_{n=0}^{\infty}(I_n) = I_0 + I_1 + I_2 + \ldots\infty$$

$$= \int dx + \int \cos\frac{x}{2}dx + \int \cos^2\frac{x}{2}dx + \ldots\infty$$

$$= \int\left(1 + \cos\frac{x}{2} + \cos^2\frac{x}{2} + \ldots\infty\right)dx$$

$$= \int \frac{1}{1 - \cos\frac{x}{2}}\,dx \qquad \left[\because -1 < \cos\frac{x}{2} < 1\right]$$

$$= \int \frac{1 + \cos\frac{x}{2}}{\sin^2\frac{x}{2}}\,dx = \int\left(\operatorname{cosec}^2\frac{x}{2} + \cot\frac{x}{2}\operatorname{cosec}\frac{x}{2}\right)dx$$

$$= -2\cot\frac{x}{2} - 2\operatorname{cosec}\frac{x}{2} + C$$

13. *(a)* Let $I = \int \tan^4 x\,dx$

$$= \int \tan^2 x(\sec^2 x - 1)\,dx$$

$$= \int \tan^2 x \sec^2 x\,dx - \int \tan^2 x\,dx$$

$$= \int \tan^2 x\,d(\tan x) - \int(\sec^2 x - 1)\,dx$$

$$= \frac{\tan^3 x}{3} - \tan x + x + C$$

$$= K\tan^3 x + L\tan x + f(x)$$ [given]

On comparing, $K = \frac{1}{3}, L = -1, \quad f(x) = x + C$

14. *(d)* $\int_0^{2x}(1 - t)f(t)\,dt = -2x^3 - \frac{x^2}{2} + 2x + K$

On differentiating both sides, we get

$(1 - 2x)f(2x)\cdot 2 - 0 = -6x^2 - x + 2$

$\Rightarrow \quad 2(1 - 2x)f(2x) = (1 - 2x)(3x + 2)$

$\Rightarrow \quad f(2x) = \frac{3}{2}x + 1$

$$\Rightarrow \quad f(x) = \frac{3}{4}x + 1$$

$$\therefore \quad \sum_{r=1}^{n} f(r) = \frac{3}{4}\sum_{r=1}^{n} r + \sum_{r=1}^{n} (1)$$

$$= \frac{3}{4} \cdot \frac{n(n+1)}{2} + n = \frac{n(3n+11)}{8}$$

$$\therefore \quad \sum_{r=1}^{8} f(r) = \frac{8 \times 35}{8} = 35$$

15. ***(c)*** Since, $xy - 3x - 2y - 10 = 0$

$$\Rightarrow \quad y = \frac{3x + 10}{x - 2}$$

$$\text{Required area} = \int_3^4 y\,dx = \int_3^4 \frac{3x + 10}{x - 2}\,dx$$

$$= \int_3^4 \frac{3(x-2) + 16}{x - 2}\,dx$$

$$= \int_3^4 \left(3 + \frac{16}{x - 2}\right)dx$$

$$= [3x + 16 \log (x - 2)]_3^4$$

$$= 3 + 16 \log 2$$

16. ***(a)*** $\frac{dx}{dy} = \frac{x^2 + y^2}{xy}$

Putting $x = vy \Rightarrow \frac{dx}{dy} = v + y\frac{dv}{dy}$

$$\therefore \quad v + y\frac{dv}{dy} = \frac{v^2 + 1}{v} = v + \frac{1}{v}$$

$$\Rightarrow \quad vdv = \frac{dy}{y}$$

On integrating both sides, we get

$$\frac{v^2}{2} = \log y + C \quad \Rightarrow \quad \frac{x^2}{2y^2} = \log y + C$$

Since, $y(1) = 1 \Rightarrow C = \frac{1}{2}$

and $y(k) = \sqrt{e} \Rightarrow \frac{k^2}{2e} = \log \sqrt{e} + C$

$$\Rightarrow \quad \frac{k^2}{2e} = \frac{1}{2} + \frac{1}{2} = 1 \Rightarrow k = \sqrt{2e}$$

17. ***(c)*** Let the equation of the family of circles be

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

It passes through $(a, 0)$ and $(-a, 0)$. Therefore,

$$a^2 + 2ga + c = 0 \qquad \ldots(i)$$

and $\quad a^2 - 2ag + c = 0 \qquad \ldots(ii)$

On solving Eqs. (i) and (ii), we get

$$c = -a^2 \text{ and } g = 0$$

Thus, equation of family of circles is

$$x^2 + y^2 + 2fy - a^2 = 0 \qquad \ldots(iii)$$

$$\Rightarrow \quad 2x + 2yy_1 + 2fy_1 = 0$$

$$\Rightarrow \quad f = -\left(\frac{x + yy_1}{y_1}\right)$$

On substituting the value of f in Eq. (iii), we get

$$y_1(y^2 - x^2 + a^2) + 2xy = 0$$

18. ***(a)*** The angle between $\mathbf{a}$ and $\mathbf{b}$ is acute.

$$\Rightarrow \quad \mathbf{a} \cdot \mathbf{b} > 0$$

$$\Rightarrow \quad 2x^2 - 3x + 1 > 0$$

$$\Rightarrow \quad (2x - 1)(x - 1) > 0$$

$$\Rightarrow \quad x < \frac{1}{2} \text{ or } x > 1 \qquad \ldots(i)$$

The angle between $\mathbf{b}$ and Y-axis lies between $\frac{\pi}{2}$ and π.

$$\therefore \quad \mathbf{b} \cdot \mathbf{j} < 0 \quad \Rightarrow \quad x < 0 \qquad \ldots(ii)$$

From Eqs. (i) and (ii), $x < 0$

19. ***(b)*** $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{a} - 2\mathbf{b})$

$$= (\mathbf{a} \times \mathbf{b}) \times \mathbf{a} - 2(\mathbf{a} \times \mathbf{b}) \times \mathbf{b}$$

$$= -\mathbf{a} \times (\mathbf{a} \times \mathbf{b}) + 2\mathbf{b} \times (\mathbf{a} \times \mathbf{b})$$

$$= -\{(\mathbf{a} \cdot \mathbf{b})\mathbf{a} - (\mathbf{a} \cdot \mathbf{a})\mathbf{b}\} + 2\{(\mathbf{b} \cdot \mathbf{b})\mathbf{a} - (\mathbf{b} \cdot \mathbf{a})\mathbf{b}\}$$

$$= -\left\{\frac{1}{2}\mathbf{a} - \mathbf{b}\right\} + 2\left\{\mathbf{a} - \frac{1}{2}\mathbf{b}\right\} = \frac{3}{2}\mathbf{a}$$

$$\therefore (2\mathbf{a} + \mathbf{b}) \cdot [(\mathbf{a} \times \mathbf{b}) \times (\mathbf{a} - 2\mathbf{b})]$$

$$= \frac{3}{2}(2\mathbf{a} + \mathbf{b}) \cdot \mathbf{a} = \frac{3}{2}(2\mathbf{a} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{a})$$

$$= \frac{3}{2}\left(2 + \frac{1}{2}\right) = \frac{15}{4}$$

20. ***(d)*** Direction ratios of the line joining points $P(1, 2, 3)$ and $Q(-3, 4, 5)$ are $-4, 2, 2$.

So, direction ratios of normal to the plane are $-4, 2, 2$.

Then, equation of plane is $-4x + 2y + 2z = k$

This plane passes through the mid-point of $PQ(-1, 3, 4)$.

$$\Rightarrow \quad -4(-1) + 2(3) + 2(4) = k \quad \Rightarrow \quad k = 18$$

$\therefore$ Equation of plane is $2x - y - z = -9$

$$\Rightarrow \quad a = -\frac{9}{2}, b = 9 \text{ and } c = 9$$

$$\therefore \quad 2a + b + c = 2\left(-\frac{9}{2}\right) + 9 + 9 = 9$$

21. ***(a)*** The equation of the plane through the intersection of the planes $x + y + z = 1$ and $2x + 3y - z + 4 = 0$ is

$$(x + y + z - 1) + \lambda(2x + 3y - z + 4) = 0$$

or $\quad (2\lambda + 1)x + (3\lambda + 1)y + (1 - \lambda)z + 4\lambda - 1 = 0 \qquad \ldots(i)$

It is parallel to X-axis, i.e. $\frac{x}{1} = \frac{y}{0} = \frac{z}{0}$

$$\therefore \quad 1(2\lambda + 1) + 0 \times (3\lambda + 1) + 0 \times (1 - \lambda) = 0$$

$$\Rightarrow \quad \lambda = -\frac{1}{2}$$

On substituting $\lambda = -\frac{1}{2}$ in Eq. (i), we get

$$y - 3z + 6 = 0$$

22. ***(b)*** Consider the following events

A_1 : A speaks truth

A_2 : B speaks truth

Then, $P(A_1) = \frac{60}{100} = \frac{3}{5}$, $P(A_2) = \frac{70}{100} = \frac{7}{10}$

For the required event, either of them should speak a truth.

Required probability

$$= P[(A_1 \cap \bar{A}_2) \cup (\bar{A}_1 \cap A_2)]$$

$$= P(A_1 \cap \bar{A}_2) + P(\bar{A}_1 \cap A_2)$$

$$= P(A_1)P(\bar{A}_2) + P(\bar{A}_1)P(A_2)$$

$$= \frac{3}{5} \times \left(1 - \frac{7}{10}\right) + \left(1 - \frac{3}{5}\right) \times \frac{7}{10}$$

$$= \frac{3}{5} \times \frac{3}{10} + \frac{2}{5} \times \frac{7}{10} = 0.46$$

23. ***(b)*** Since, $P(A \cap B) = \frac{1}{4} \neq 0$

Therefore, A and B are not mutually exclusive events.

Now, $P(\overline{A \cup B}) = \frac{1}{6}$

$\Rightarrow 1 - P(A \cup B) = \frac{1}{6}$

$\Rightarrow P(A \cup B) = \frac{5}{6}$

$\Rightarrow P(A) + P(B) - P(A \cap B) = \frac{5}{6}$

$\Rightarrow P(B) = \frac{5}{6} + \frac{1}{4} - \frac{3}{4} = \frac{1}{3}$

Clearly, $P(A \cap B) = \frac{1}{4} = \frac{3}{4} \times \frac{1}{3} = P(A) \times P(B)$

So, A and B are independent.

Also, $P(A) \neq P(B) \Rightarrow A$ and B are not equally likely.

24. ***(a)*** $y^2 = \alpha x^3 - \beta$

$\Rightarrow 2y\frac{dy}{dx} = 3\alpha x^2 \Rightarrow \frac{dy}{dx} = \frac{3\alpha x^2}{2y}$

$\Rightarrow \left(\frac{dy}{dx}\right)_{(2,3)} = \frac{12\alpha}{6} = 2\alpha$

Slope of normal $= -\frac{1}{2\alpha} = -\frac{1}{4}$

$\Rightarrow \alpha = 2$

Also, (2, 3) lies on the curve.

$9 = 8\alpha - \beta \Rightarrow \beta = 16 - 9 = 7$

$\therefore$ Required equation is $x^2 - (7 + 2)x + 7 \times 2 = 0$

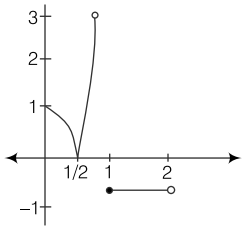
or $x^2 - 9x + 14 = 0$

25. ***(c)*** Since, $1 \le x < 2 \Rightarrow 0 \le x - 1 < 1$

$\Rightarrow [x^2 - 2x] = [(x-1)^2 - 1] = -1$

$\therefore f(x) = \begin{cases} 1 - 4x^2, & 0 \le x < \frac{1}{2} \\ 4x^2 - 1, & \frac{1}{2} \le x < 1 \\ -1, & 1 \le x < 2 \end{cases}$

$\therefore$ Graph of $f(x)$ is given by



It is clear from graph that $f(x)$ is discontinuous at $x = 1$ and not differentiable at $x = 1$ and $x = \frac{1}{2}$.

26. ***(a)*** **Reflexivity** Let a be an arbitrary element of R.

Then, $a \in R$

$\Rightarrow 1 + a \cdot a = 1 + a^2 > 0 \quad [\because a^2 > 0, \forall\, a \in R]$

$\Rightarrow (a, a) \in R_1$

So, R_1 is reflexive on R.

Symmetry Let $(a, b) \in R$, then

$(a, b) \in R_1$

$\Rightarrow 1 + ab > 0$

$\Rightarrow 1 + ba > 0$

$\Rightarrow (b, a) \in R_1$

So, R_1 is symmetric on R.

Transitivity We observe that $\left(1, \frac{1}{2}\right) \in R_1$ and $\left(\frac{1}{2}, -1\right) \in R_1$ but $(1, -1) \notin R_1$ because $1 + (1)(-1) = 0$

So, R_1 is not transitive on R.

27. ***(d)*** Let $f(x) = \frac{|\sin x + \cos x|}{|\sin x| + |\cos x|}$

$\therefore f(\pi + x) = \frac{|\sin(\pi + x) + \cos(\pi + x)|}{|\sin(\pi + x)| + |\cos(\pi + x)|}$

$= \frac{|-(\sin x + \cos x)|}{|-\sin x| + |-\cos x|} = \frac{|\sin x + \cos x|}{|\sin x| + |\cos x|} = f(x)$

So, $f(x)$ is a periodic function with a period of π.

$\therefore \int_a^{a+\pi} \frac{|\sin x + \cos x|}{|\sin x| + |\cos x|}\, dx$ is independent of a.

28. ***(c)*** $\vec{\alpha} + \vec{\beta} + \vec{\gamma} = a\vec{\delta}$

$\Rightarrow \vec{\alpha} + \vec{\beta} + \vec{\gamma} + \vec{\delta} = (a + 1)\vec{\delta}$ ...(i)

Also, $\vec{\beta} + \vec{\gamma} + \vec{\delta} = b\vec{\alpha}$

$\Rightarrow \vec{\alpha} + \vec{\beta} + \vec{\gamma} + \vec{\delta} = (b + 1)\vec{\alpha}$ ...(ii)

From Eqs. (i) and (ii), we get

$(a + 1)\vec{\delta} = (b + 1)\vec{\alpha}$

Since, $\vec{\alpha}$ is not parallel to $\vec{\delta}$.

$\therefore \quad a + 1 = 0$ and $b + 1 = 0$

From Eq. (ii), $\vec{\alpha} + \vec{\beta} + \vec{\gamma} + \vec{\delta} = 0$

29. ***(a)*** Let l, m, n be the direction cosines.

Then, $l = \cos\theta$, $m = \cos\beta$ and $n = \cos\theta$

$\therefore \quad l^2 + m^2 + n^2 = 1$

$\Rightarrow \cos^2\theta + \cos^2\beta + \cos^2\theta = 1$

$\Rightarrow 2\cos^2\theta + 1 - \sin^2\beta = 1$

$\Rightarrow 2\cos^2\theta - 2\sin^2\theta = 0 \quad [\because \sin^2\beta = 2\sin^2\theta]$

$\Rightarrow 2\cos^2\theta - 2(1 - \cos^2\theta) = 0$

$\Rightarrow 4\cos^2\theta - 2 = 0$

$\Rightarrow \cos\theta = \pm\frac{1}{\sqrt{2}} \Rightarrow \sin\theta = \pm\frac{1}{\sqrt{2}}$

$\therefore \quad \cos\beta = \sqrt{1 - \sin^2\beta} = \sqrt{1 - 2\sin^2\theta} = \sqrt{1 - 1} = 0$

$\therefore$ Direction cosines are $\pm\frac{1}{\sqrt{2}}, 0, \pm\frac{1}{\sqrt{2}}$.

30. ***(b)*** It is given that x, y, z are not all zero.

So, the given system of equations has a non-trivial solution.

$\therefore \begin{vmatrix} 1 & -a & a \\ b & 1 & -b \\ c & -c & -1 \end{vmatrix} = 0$

$\Rightarrow 1[-1 - bc] + a[-b + bc] + a[-bc - c] = 0$

$\Rightarrow -1 - bc - ab + abc - abc - ac = 0$

$\Rightarrow 1 + ab + bc + ca = 0$

Now, $a + b + c > 1$

$\Rightarrow (a + b + c)^2 > 1$

$\Rightarrow a^2 + b^2 + c^2 + 2(ab + bc + ca) > 1$

$\Rightarrow a^2 + b^2 + c^2 > 3$

So, both the Statements are true but Statement II is not correct explanation of Statement I.

TARGET JEE 2017

TEST RIDER

Comprehensive Simulator Test Series for JEE Main & Advanced

JEE MAIN

(PAPER TWO SCALE UP)

HALF TEST 2

(XII SYLLABUS)

WITH ANSWER KEY

Mock Questions from Class XIIth syllabus to enhance Your Problem Solving Skills

INSTRUCTIONS

- This test consists of 30 questions and each question is allotted 4 marks for correct response.
- Candidates will be awarded marks as stated above for correct response of each question. 1/4 mark will be deducted for indicating incorrect response of each question. No deduction from the total score will be made if no response is indicated for an item in the answer sheet.
- There is only one correct response for each question. Filling up more than one response in any question will be treated as wrong response and marks for wrong response will be deducted according as per instructions.

1. The integral $\int x\,\text{cosec}^{-1}\left(\frac{2\sqrt{a}}{\sqrt{2a-x}}\right)dx$ equals

(a) $\frac{1}{8}\left[2(x^2-2a^2)\cos^{-1}\frac{x}{2a}+x\sqrt{4a^2-x^2}\right]+C$

(b) $\frac{1}{8}\left[2(x^2-2a^2)\cos^{-1}\frac{x}{2a}-x\sqrt{4a^2-x^2}\right]+C$

(c) $\frac{1}{8}\left[2(x^2+2a^2)\cos^{-1}\frac{x}{2a}+x\sqrt{4a^2-x^2}\right]+C$

(d) $\frac{1}{8}\left[2(x^2-2a^2)\cos^{-1}\frac{x}{2a}-x\sqrt{4a^2+x^2}\right]+C$

2. The number of positive integral solutions of the equation $\tan^{-1}x+\sec^{-1}\left(\frac{\sqrt{1-y^2}}{y}\right)=\sin^{-1}\frac{3}{\sqrt{10}}$ is

(a) 4 (b) 3 (c) 2 (d) 1

3. The integral $\int_0^{\pi}x^2\{(1+\sin x)^{-2}\cos x\}\,dx$ is equal to

(a) $\pi^2+2\pi$ (b) $\pi^2-2\pi$ (c) $-\pi^2+2\pi$ (d) $-\pi^2-2\pi$

4. The integral $\int_{\pi/6}^{\pi/3}\frac{f'(x)}{f(x)+f\left(\frac{\pi}{2}-\pi\right)}dx$ is equal to

(a) $\pi/12$ (b) $12/\pi$ (c) $\pi/13$ (d) $13/\pi$

5. If m is odd and n is an even integer, then the integral $\int_0^{\pi}\sin(m\theta)\cos(n\theta)d\theta$ is equal to

(a) $\frac{2n}{n^2-m^2}$ (b) $\frac{2m}{m^2-n^2}$ (c) $\frac{m^2+n^2}{m^2-n^2}$ (d) $\frac{m^2+n^2}{n^2-m^2}$

6. The area bounded by the curves $y^2=4a^2(x-2)$ and lines $x=2$ and $y=4a$ is

(a) $4a^2$ sq units (b) $5a^2$ sq units

(c) $(16a/3)$ sq units (d) $(16a^2/3)$ sq units

7. $\lim_{n\to\infty}\left\{\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{6n}\right\}$ is equal to

(a) log 4 (b) log 5 (c) log 6 (d) log 7

8. Area bounded by the curves $y=|x-2|$ and $y=2$ is equal to

(a) 2 sq units (b) 3 sq units (c) 4 sq units (d) 6 sq units

9. The integral $\int_0^{\pi/2}\frac{\sin^2 x}{\sin x+\cos x}dx$ equals

(a) $\frac{1}{\sqrt{2}}\log(\sqrt{2}-1)$ (b) $\frac{1}{\sqrt{2}}\log(\sqrt{2}+1)$

(c) $\frac{1}{\sqrt{3}}\log(\sqrt{2}+1)$ (d) $\frac{1}{\sqrt{3}}\log(\sqrt{2}-1)$

10. If $a=2^n-1$, $b=\frac{3^n-1}{2}$ and $c=\frac{5^n-1}{4}$, then $\begin{vmatrix} a & 2b & 4c \\ 2 & 2 & 2 \\ 2^n & 3^n & 5^n \end{vmatrix}$ equals

(a) 0 (b) $(10)^n$ (c) $(20)^n$ (d) $(30)^n$

11. If ω is cube root of unity and $x_1+x_2+x_3=a$ $x_1+\omega x_2+\omega^2 x_3=b$ and $x_1+\omega^2 x_2+\omega x_3=c$, then which of the following is correct?

(a) $x_1=\frac{a+2b+3c}{3}$ (b) $x_2=\frac{a+b\omega+c\omega^2}{3}$

(c) $x_3=\frac{a+b\omega+c\omega^2}{3}$ (d) None of these

12. If $y=(k+lx)\cos px+(m+nx)\sin px$ and $\frac{d^4y}{dx^4}+2p^2\frac{d^2y}{dx^2}-ty=0$, then t equals

(a) p^3 (b) p^{-3} (c) p^4 (d) p^{-4}

13. If a_1, a_2, a_3 are non-zeroes, then the system of equations $(\alpha + a_1)x_1 + \alpha x_2 + \alpha x_3 = 0$, $\alpha x_1 + (\alpha + a_2)x_2 + \alpha x_3 = 0$, $\alpha x_1 + \alpha x_2 + (\alpha + a_3)x_3 = 0$ has a non-trivial solution, if

(a) $\alpha^{-1} = a_1 + a_2 + a_3$ *(b)* $\alpha^{-1} = -(a_1^{-1} + a_2^{-1} + a_3^{-1})$

(c) $\alpha + a_1 + a_2 + a_3 = 2$ *(d)* $2\alpha - a_1 - a_2 - a_3 = 0$

14. If $f(x) = \begin{cases} x^2(x-1) + 5(2x-1), & x \le 1 \\ -2x + \log_2(a^2 - 2), & x > 1 \end{cases}$

has greatest value at $x = 1$, then the set of value of a equals

(a) $(-\sqrt{3}, \sqrt{3})$ *(b)* $(\sqrt{2}, \sqrt{130}]$

(c) $[-\sqrt{130}, \sqrt{130}]$ *(d)* $(\sqrt{3}, \sqrt{130}]$

15. Let $f(x)$ be a function such that

$f(x) = \begin{cases} x^a \sin\dfrac{1}{x}, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \end{cases}$, where a is a constant. If $f(x)$ is a derivable, $\forall\, x > 0$, then least integral value of a is

(a) 1 *(b)* 3 *(c)* 2 *(d)* 4

16. If $f(x)$ satisfies the condition for Rolle's theorem in $[5, 7]$, then $\int_5^7 f(x)\,dx$ equals

(a) $\dfrac{140}{3}$ *(b)* $\dfrac{146}{3}$ *(c)* $\dfrac{160}{3}$ *(d)* $\dfrac{155}{3}$

17. If $f(x) = \begin{cases} e^{-3x}, \text{ if } x \ge 0 \\ e^{3x}, \text{ if } x < 0 \end{cases}$, then $f(x)$ satisfies the condition

(a) continuous, $\forall x \in R$

(b) differentiable at $x = 0$

(c) continuous, $\forall\, x \in R$ and differentiable at $x = 0$

(d) not continuous, $\forall x \in R$ and non-differentiable at $x = 0$

18. The minimum value of $f(x) = x^2 - 2x + 2 + |\sin x|$ equals

(a) 0 *(b)* –1 *(c)* 2 *(d)* 1

19. If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are three unit vectors such that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \dfrac{\sqrt{3}}{2}\mathbf{b}$, where $\mathbf{b}$ and $\mathbf{c}$ being non-parallel. If θ_1 is the angle between $\mathbf{a}$ and $\mathbf{b}$, and θ_2 is the angle between $\mathbf{a}$ and $\mathbf{c}$, then

(a) $\theta_1 = \dfrac{\pi}{6}, \theta_2 = \dfrac{\pi}{2}$ *(b)* $\theta_1 = \dfrac{\pi}{3}, \theta_2 = \dfrac{\pi}{4}$

(c) $\theta_1 = \dfrac{\pi}{2}, \theta_2 = \dfrac{\pi}{6}$ *(d)* $\theta_1 = \dfrac{\pi}{4}, \theta_2 = \dfrac{\pi}{3}$

20. The equation of the curve passing through the origin and satisfying the differential equation $y' = \sin(10x + 6y)$

(a) $\tan(5x + 3y) = \dfrac{1}{5\tan 4x}$

(b) $\tan(5x + 3y) = \dfrac{5\tan 4x}{4 - 3\tan 4x}$

(c) $4 + 3\tan 4x = \tan(5x + 3y) + xy$

(d) $\tan(5x - 3y) = \dfrac{5\tan 4x}{4 + 3\tan 4x}$

21. $[\mathbf{a} \times (3\mathbf{b} + 2\mathbf{c})\ \mathbf{b} \times (\mathbf{c} - 2\mathbf{a})\ 2\mathbf{c} \times (\mathbf{a} - 3\mathbf{b})]$ equals

(a) $-18[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2$ *(b)* $18[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2$

(c) $5[\mathbf{a} \times \mathbf{b}\ \mathbf{b} \times \mathbf{c}\ \mathbf{c} \times \mathbf{a}]$ *(d)* $-5[\mathbf{a} \times \mathbf{b}\ \mathbf{b} \times \mathbf{c}\ \mathbf{c} \times \mathbf{a}]$

22. The differential equation of the family of circles passing the origin and $(c, 0)$ is

(a) $(x^2 - y^2 + cx)\dfrac{dy}{dx} = (2x - c)\,y$

(b) $(x^2 - y^2 - cx)\dfrac{dy}{dx} = (2x - c)\,y$

(c) $(x^2 + y^2 + cx)\dfrac{dy}{dx} = (2x - c)\,y$

(d) $(x^2 - y^2 - cx)\dfrac{dy}{dx} = (2x + c)\,y$

23. If $\mathbf{x} + \mathbf{c} \times \mathbf{y} = \mathbf{a}$ and $\mathbf{y} + \mathbf{c} \times \mathbf{x} = \mathbf{b}$, where $\mathbf{c}$ is a non-zero vector, then $\mathbf{x}$ equals

(a) $\dfrac{\mathbf{c} \times \mathbf{b} + \mathbf{b} + (\mathbf{c} \cdot \mathbf{a})\,\mathbf{c}}{1 + \mathbf{c} \cdot \mathbf{c}}$ *(b)* $\dfrac{\mathbf{c} \times \mathbf{b} + \mathbf{c} + (\mathbf{a} \cdot \mathbf{b})\,\mathbf{c}}{1 + \mathbf{c} \cdot \mathbf{c}}$

(c) $\dfrac{\mathbf{a} \times \mathbf{c} + \mathbf{b} + (\mathbf{b} \cdot \mathbf{c})\,\mathbf{b}}{1 + \mathbf{c} \cdot \mathbf{c}}$ *(d)* $\dfrac{\mathbf{c} \times \mathbf{a} + \mathbf{a} + (\mathbf{c} \times \mathbf{b})\,\mathbf{c}}{1 + \mathbf{c} \cdot \mathbf{c}}$

24. If line $\dfrac{x-2}{3} = \dfrac{y+1}{2} = \dfrac{z-1}{-1}$ intersect to the curve $xy^2 = c^2$, $z = 0$, then c equals

(a) $\pm\sqrt{2}$ *(b)* $\pm\sqrt{3}$ *(c)* $\pm\sqrt{5}$ *(d)* $\pm\sqrt{6}$

25. The plane $ax + by = cz = d$, meets the coordinate axes at the points P, Q and R respectively. Area of ΔPQR is equal to

(a) $\dfrac{d^2\sqrt{a^2 + b^2 + c^2}}{2|abc|}$ *(b)* $\dfrac{d^2\sqrt{a^2 + b^2 + c^2}}{4|abc|}$

(c) $\dfrac{d^2\sqrt{a^2 + b^2 + c^2}}{|abc|}$ *(d)* $\dfrac{d^2\sqrt{a^2 + b^2 + c^2}}{3|abc|}$

26. Two dice are thrown n times in succession. The probability of obtaining a double five atleast once is

(a) $\left(\dfrac{1}{12}\right)^n$ *(b)* $1 - \left(\dfrac{35}{36}\right)^n$ *(c)* $\left(\dfrac{1}{36}\right)^n$ *(d)* $\left(\dfrac{35}{36}\right)^n$

27. If the distance between the plane $x - 2y + z = d$ and the plane containing the lines $\dfrac{x-1}{2} = \dfrac{y-2}{3} = \dfrac{z-3}{4}$ and $\dfrac{x-2}{3} = \dfrac{y-4}{4} = \dfrac{z-4}{5}$ is $\sqrt{6}$, then $|d|$ is equal to

(a) 5 *(b)* 1 *(c)* 4 *(d)* 6

28. Let U be the universal set and $n(X) = p + 1$, the probability of selecting 2 subsets P and Q of the set X such that $Q = P'$, equals

(a) $\dfrac{2}{2^k - 1}$ *(b)* $\dfrac{3}{2^k - 1}$ *(c)* $\dfrac{1}{2 \cdot 2^k - 1}$ *(d)* $\dfrac{1}{2^k - 1}$

29. The equation of line of intersection of planes $4x + 4y - 5z = 12$, $8x + 12y - 13z = 32$ can be written as

(a) $\dfrac{x-1}{2} = \dfrac{y-2}{3} = \dfrac{z}{4}$ *(b)* $\dfrac{x}{2} = \dfrac{y}{3} = \dfrac{z-2}{4}$

(c) $\dfrac{x-1}{2} = \dfrac{y+2}{-3} = \dfrac{z}{4}$ *(d)* $\dfrac{x}{2} = \dfrac{y+1}{3} = \dfrac{z-2}{4}$

30. A coin is tossed 10 times. The probability of getting an even number of heads equals

(a) 1/3 *(b)* 1/2 *(c)* 1/4 *(d)* 2/3

Answers

1. (b)	**2.** (c)	**3.** (c)	**4.** (a)	**5.** (b)
6. (c)	**7.** (a)	**8.** (c)	**9.** (b)	**10.** (a)
11. (c)	**12.** (d)	**13.** (b)	**14.** (b)	**15.** (c)
16. (b)	**17.** (a)	**18.** (d)	**19.** (c)	**20.** (b)
21. (a)	**22.** (b)	**23.** (a)	**24.** (c)	**25.** (a)
26. (b)	**27.** (d)	**28.** (c)	**29.** (a)	**30.** (b)

TRIGONOMETRY

A collection of best questions asked in JEE to revise your concepts

1. If $\theta = \frac{2\pi}{2009}$, then $\cos\theta \cos 2\theta \cos 3\theta \ldots \cos 1004\theta$ is [2015]

(a) 0 (b) $\frac{1}{2^{2008}}$ (c) $\frac{1}{2^{1004}}$ (d) $-\frac{1}{2^{1004}}$

2. The positive integral value of $n > 3$ satisfying $\frac{1}{\sin\left(\frac{\pi}{n}\right)} = \frac{1}{\sin\left(\frac{2\pi}{n}\right)} + \frac{1}{\sin\left(\frac{3\pi}{n}\right)}$ is [2011]

(a) 6 (b) 7 (c) 8 (d) 9

3. If $\frac{\sin^4 x}{2} + \frac{\cos^4 x}{3} = \frac{1}{5}$, then [2009]

(a) $\tan^2 x = \frac{2}{3}$ (b) $\frac{\sin^8 x}{8} + \frac{\cos^8 x}{27} = \frac{1}{125}$

(c) $\tan^2 x = \frac{1}{3}$ (d) $\frac{\sin^8 x}{8} + \frac{\cos^8 x}{27} = \frac{2}{125}$

4. In a triangle, the sum of two sides is x and the product of the same side is y. If $x^2 - c^2 = y$, where c is the third side of the triangle, then the ratio of the inradius to the circumradius of the triangle is [2014]

(a) $\frac{3y}{2x\ (x + c)}$ (b) $\frac{3y}{2c\ (x + c)}$

(c) $\frac{3y}{4x\ (x + c)}$ (d) $\frac{3y}{4c\ (x + c)}$

5. In a ΔPQR, P is the largest angle and $\cos P = \frac{1}{3}$. Further the incircle of the triangle touches the sides PQ, QR and RP at N, L and M respectively, such that the lengths of PN, QL and RM are consecutive even integers. Then, possible length(s) of the side(s) of the triangle is(are) [2013]

(a) 16, 18 (b) 18, 22 (c) 22, 24 (d) 16, 20

6. Let PQR be a triangle of area Δ with $a = 2$, $b = \frac{7}{2}$ and $c = \frac{5}{2}$, where a, b and c are the lengths of the sides of the triangle opposite to the angles P, Q and R respectively. Then, $\frac{2\sin P - \sin 2P}{2\sin P + \sin 2P}$ equals [2012]

(a) $\frac{3}{4\Delta}$ (b) $\frac{45}{4\Delta}$ (c) $\left(\frac{3}{4\Delta}\right)^2$ (d) $\left(\frac{45}{4\Delta}\right)^2$

7. Sides a, b, c of ΔABC are in AP and $\cos\theta_1 = \frac{a}{b+c}$, $\cos\theta_2 = \frac{b}{a+c}$, $\cos\theta_3 = \frac{c}{a+b}$, then $\tan^2\frac{\theta_1}{2} + \tan^2\frac{\theta_3}{2}$ is equal to [2006]

(a) $\frac{2}{3}$ (b) 1

(c) $\frac{\sqrt{5}}{3}$ (d) None of these

8. In ΔABC, if $\frac{\sin A}{c\sin B} + \frac{\sin B}{c} + \frac{\sin C}{b} = \frac{c}{ab} + \frac{b}{ac} + \frac{a}{bc}$, then the value of A is [2015]

(a) 120° (b) 90° (c) 60° (d) 30°

9. $ABCD$ is a trapezium such that AB and CD are parallel and $BC \perp CD$. If $\angle ADB = \theta$, $BC = p$ and $CD = q$, then AB is equal to [2013]

(a) $\frac{(p^2 + q^2)\sin\theta}{p\cos\theta + q\sin\theta}$ (b) $\frac{p^2 + q^2\cos\theta}{p\cos\theta + q\sin\theta}$

(c) $\frac{p^2 + q^2}{p^2\cos\theta + q^2\sin\theta}$ (d) $\frac{(p^2 + q^2)\sin\theta}{(p\cos\theta + q\sin\theta)^2}$

10. The number of distinct solutions of the equation $\frac{5}{4}\cos^2 2x + \cos^4 x + \sin^4 x + \cos^6 x + \sin^6 x = 2$ in the interval $[0, 2\pi]$ is [2015]

(a) 4 (b) 6
(c) 8 (d) 5

11. For $x \in (0, \pi)$, the equation $\sin x + 2\sin 2x - \sin 3x = 3$ has [2014]

(a) infinitely solutions
(b) three solutions
(c) one solution
(d) no solution

12. The number of all possible values of θ, where $0 < \theta < \pi$, for which the system of equations

$$(y + z)\cos 3\theta = xyz\sin 3\theta$$

$$x\sin 3\theta = \frac{2\cos 3\theta}{y} + \frac{2\sin 3\theta}{z}$$

$$xyz\sin 3\theta = (y + 2z)\cos 3\theta + y\sin 3\theta$$

have a solution (x_0, y_0, z_0) with $y_0 z_0 \neq 0$, is [2010]

(a) 0 (b) 2
(c) 3 (d) 4

13. The set of values of θ satisfying the inequation $2\sin^2\theta - 5\sin\theta + 2 > 0$, where $0 < \theta < 2\pi$, is [2006]

(a) $\left(0, \frac{\pi}{6}\right) \cup \left(\frac{5\pi}{6}, 2\pi\right)$ (b) $\left[0, \frac{\pi}{6}\right] \cup \left[\frac{5\pi}{6}, 2\pi\right]$

(c) $\left[0, \frac{\pi}{3}\right] \cup \left[\frac{2\pi}{3}, 2\pi\right]$ (d) None of these

14. For $0 < \theta < \frac{\pi}{2}$, the solution(s) of

$$\sum_{m=1}^{6} \operatorname{cosec}\left(\theta + (m-1)\frac{\pi}{4}\right)\operatorname{cosec}\left(\theta + \frac{m\pi}{4}\right) = 4\sqrt{2}$$

is(are) [2009]

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{6}$

(c) $\frac{\pi}{12}$ (d) $\frac{5\pi}{12}$

15. If the angles of elevation of the top of a tower from three collinear points A, B and C, on a line leading to the foot of the tower are 30°, 45° and 60°, respectively, then the ratio $AB : BC$, is [2015]

(a) $1 : \sqrt{3}$ (b) $2 : 3$ (c) $\sqrt{3} : 1$ (d) $\sqrt{3} : \sqrt{2}$

16. A bird is sitting on the top of a vertical pole 20 m high and its elevation from a point O on the ground is 45°. If flies off horizontally straight away from the point O. After one second, the elevation of the bird from O is reduced to 30°. Then, the speed (in m/s) of the bird is [2014]

(a) $20\sqrt{2}$ (b) $20(\sqrt{3} - 1)$

(c) $40(\sqrt{2} - 1)$ (d) $40(\sqrt{3} - \sqrt{2})$

17. A circular ring of radius 3 cm is suspended horizontally from a point 4 cm vertically above the centre by 4 strings attached at equal intervals to its circumference. If the angle between two consecutive strings be θ, then $\cos\theta$ is equal to [2007]

(a) $\frac{4}{5}$ (b) $\frac{4}{25}$

(c) $\frac{16}{25}$ (d) None of these

18. In the ambiguous case, if a, b and A are given and c_1, c_2 are the two values of the third side, then $(c_1 - c_2)^2 + (c_1 + c_2)^2 \tan^2 A$ is equal to [2009]

(a) 4 (b) $4a^2$

(c) $4b^2$ (d) $4c^2$

19. The internal bisector of $\angle A$ of ΔABC meets side BC at D. A line drawn through D perpendicular to AD intersects the side AC at E and side AB at F. If a, b, c represent the lengths of the sides BC, CA and AB respectively, then [2011]

(a) $AD = \frac{2bc}{b+c}\cos\frac{A}{2}$ (b) AE is HM of b and c

(c) $EF = \frac{4bc}{b+c}\sin\frac{A}{2}$ (d) All of these

20. The maximum value of the expression $\frac{1}{\sin^2\theta + 3\sin\theta\cos\theta + 5\cos^2\theta}$, is [2008]

(a) 2 (b) 3

(c) 4 (d) 6

Answers with Explanation

1. ***(c)*** Let

$P = \cos\theta\cos 2\theta\cos 3\theta \ldots \cos 1004\theta$

and $Q = \sin\theta\sin 2\theta\sin 3\theta \ldots \sin 1004\theta$

Then, $2^{1004}\,PQ = \sin 2\theta\sin 4\theta\sin 6\theta \ldots \sin 2008\theta$

$\Rightarrow 2^{1004}\,PQ = (\sin 2\theta\sin 4\theta\sin 6\theta \ldots \sin 1004\theta)$

$\times (\sin 1006\theta\sin 1008\theta \ldots \sin 2008\theta)$

$\Rightarrow 2^{1004}\,PQ = (\sin 2\theta\sin 4\theta\sin 6\theta \ldots \sin 1004\theta)$

$\times [\sin(2\pi - 1003\theta)\sin(2\pi - 1001\theta)\ldots\sin(2\pi - \theta)]$

$\Rightarrow 2^{1004}\,PQ = (-1)^{502}\sin\theta\sin 2\theta\sin 3\theta$

$\sin 4\theta \ldots \sin 1003\theta\sin 1004\theta$

$\Rightarrow 2^{1004}\,PQ = Q \Rightarrow P = \frac{1}{2^{1004}}$

2. ***(b)*** We have,

$$\frac{1}{\sin\left(\frac{\pi}{n}\right)} = \frac{1}{\sin\left(\frac{2\pi}{n}\right)} + \frac{1}{\sin\left(\frac{3\pi}{n}\right)}$$

$$\Rightarrow \frac{1}{\sin\left(\frac{\pi}{n}\right)} - \frac{1}{\sin\left(\frac{3\pi}{n}\right)} = \frac{1}{\sin\left(\frac{2\pi}{n}\right)}$$

$$\Rightarrow \frac{\sin\left(\frac{3\pi}{n}\right) - \sin\left(\frac{\pi}{n}\right)}{\sin\left(\frac{\pi}{n}\right)\sin\left(\frac{3\pi}{n}\right)} = \frac{1}{\sin\left(\frac{2\pi}{n}\right)}$$

$$\Rightarrow \frac{2\sin\left(\frac{\pi}{n}\right)\cos\left(\frac{2\pi}{n}\right)}{\sin\left(\frac{\pi}{n}\right)\sin\left(\frac{3\pi}{n}\right)} = \frac{1}{\sin\left(\frac{2\pi}{n}\right)}$$

$$\Rightarrow 2\sin\left(\frac{2\pi}{n}\right)\cos\left(\frac{2\pi}{n}\right) = \sin\left(\frac{3\pi}{n}\right)$$

$$\Rightarrow \sin\left(\frac{4\pi}{n}\right) = \sin\left(\frac{3\pi}{n}\right)$$

$$\Rightarrow \frac{4\pi}{n} + \frac{3\pi}{n} = \pi \Rightarrow \frac{7}{n} = 1 \Rightarrow n = 7$$

3. ***(a, b)*** We have, $\frac{\sin^4 x}{2} + \frac{\cos^4 x}{3} = \frac{1}{5}$

$$\Rightarrow \left(\frac{2+3}{2}\right)\sin^4 x + \left(\frac{2+3}{3}\right)\cos^4 x = 1$$

$$\Rightarrow \left(1 + \frac{3}{2}\right)\sin^4 x + \left(1 + \frac{2}{3}\right)\cos^4 x = (\sin^2 x + \cos^2 x)^2$$

$$\Rightarrow \left(\sqrt{\frac{3}{2}}\sin^2 x\right)^2 + \left(\sqrt{\frac{2}{3}}\cos^2 x\right)^2 - 2\sin^2 x\cos^2 x = 0$$

$$\Rightarrow \left(\sqrt{\frac{3}{2}}\sin^2 x - \sqrt{\frac{2}{3}}\cos^2 x\right)^2 = 0$$

$$\Rightarrow \sqrt{\frac{3}{2}}\sin^2 x = \sqrt{\frac{2}{3}}\cos^2 x$$

$$\Rightarrow \tan^2 x = \frac{2}{3} \text{ and } \frac{\sin^2 x}{2} = \frac{\cos^2 x}{3}$$

Now, $\frac{\sin^2 x}{2} = \frac{\cos^2 x}{3}$

$$\Rightarrow \frac{\sin^2 x}{2} = \frac{\cos^2 x}{3} = \frac{\sin^2 x + \cos^2 x}{2+3}$$

$$\Rightarrow \sin^2 x = \frac{2}{5} \text{ and } \cos^2 x = \frac{3}{5}$$

$$\Rightarrow (\sin^2 x)^4 = \frac{16}{625} \text{ and } (\cos^2 x)^4 = \frac{81}{625}$$

$$\Rightarrow \frac{\sin^8 x}{8} + \frac{\cos^8 x}{27} = \frac{2}{625} + \frac{3}{625} = \frac{1}{125}$$

4. ***(b)*** Let the lengths of two sides of the triangle, whose third side is of length c be a and b, respectively.

It is given that

$$a + b = x,\ ab = y \text{ and } x^2 - c^2 = y$$

$$\Rightarrow (a+b)^2 - c^2 = ab \Rightarrow a^2 + b^2 - c^2 = -ab$$

$$\Rightarrow \frac{a^2+b^2-c^2}{2ab} = -\frac{1}{2} \Rightarrow \cos C = -\frac{1}{2} \Rightarrow C = \frac{2\pi}{3}$$

Let Δ be the area of ΔABC. Then,

$$\Delta = \frac{1}{2}ab\sin C \Rightarrow \Delta = \frac{1}{2}y\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{4}y$$

Let r and R be respectively the inradius and circumradius of ΔABC. Then,

$$r = \frac{\Delta}{s} \text{ and } R = \frac{abc}{4\Delta} \Rightarrow \frac{r}{R} = \frac{4\Delta^2}{s(abc)}$$

$$\Rightarrow \frac{r}{R} = \frac{4\left(\frac{\sqrt{3}}{4}y\right)^2}{\left(\frac{x+c}{2}\right)yc} \qquad \left[\because s = \frac{a+b+c}{2} = \frac{x+c}{2}\right]$$

$$\Rightarrow \frac{r}{R} = \frac{3y}{2c(x+c)}$$

5. ***(b)*** Let $PN = n$, $QL = n+2$ and $RM = n+4$. Then, $PM = PN = n, QN = QL = n+2$ and $RL = RM = n+4$

$\therefore$ $PQ = n + (n+2) = 2n+2$

$QR = (n+2) + (n+4) = 2n+6$

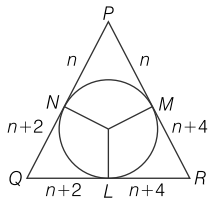
and $RP = (n+4) + n = 2n+4$

$$\therefore \cos P = \frac{PQ^2 + PR^2 - QR^2}{2PQ \cdot PR}$$

$$\Rightarrow \cos P = \frac{(2n+2)^2 + (2n+4)^2 - (2n+6)^2}{2(2n+2)(2n+4)}$$

$$= \frac{4n^2 - 16}{8(n+1)(n+2)}$$

But, $\cos P = \frac{1}{3}$



$$\therefore \frac{4(n^2-4)}{8(n+1)(n+2)} = \frac{1}{3} \Rightarrow n = 8$$

$\therefore$ $PQ = 18$, $QR = 22$ and $RP = 20$

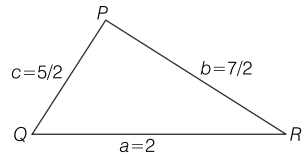
6. ***(c)*** We have, $a = 2, b = \frac{7}{2}$ and $c = \frac{5}{2}$

$$\therefore 2s = a + b + c \Rightarrow 2s = 2 + \frac{7}{2} + \frac{5}{2} \Rightarrow s = 4$$

Now,

$$\frac{2\sin P - \sin 2P}{2\sin P + \sin 2P} = \frac{2\sin P - 2\sin P\cos P}{2\sin P + 2\sin P\cos P}$$

$$= \frac{1-\cos P}{1+\cos P} = \tan^2\frac{P}{2} = \frac{(s-b)(s-c)}{s(s-a)} = \frac{(s-b)^2(s-c)^2}{\Delta^2}$$



$$= \left[\frac{(s-b)(s-c)}{\Delta}\right]^2 = \left[\frac{(2s-2b)(2s-2c)}{4\Delta}\right]^2$$

$$= \left[\frac{(8-7)(8-5)}{4\Delta}\right]^2 = \left(\frac{3}{4\Delta}\right)^2$$

7. ***(a)*** We have,

$$\cos\theta_1 = \frac{a}{b+c} \text{ and } \cos\theta_3 = \frac{c}{a+b}$$

$$\therefore \tan^2\frac{\theta_1}{2} = \frac{1-\cos\theta_1}{1+\cos\theta_1} \text{ and } \tan^2\frac{\theta_3}{2} = \frac{1-\cos\theta_3}{1+\cos\theta_3}$$

$$\Rightarrow \tan^2\frac{\theta_1}{2} = \frac{b+c-a}{a+b+c} \text{ and } \tan^2\frac{\theta_3}{2} = \frac{a+b-c}{a+b+c}$$

$$\Rightarrow \tan^2\frac{\theta_1}{2} + \tan^2\frac{\theta_3}{2} = \frac{2b}{a+b+c}$$

$$\Rightarrow \tan^2\frac{\theta_1}{2} + \tan^2\frac{\theta_3}{2} = \frac{2b}{2b+b}$$

$[\because a, b, c \text{ are in AP} \therefore a + c = 2b]$

$$\Rightarrow \tan^2\frac{\theta_1}{2} + \tan^2\frac{\theta_3}{2} = \frac{2}{3}$$

8. ***(b)*** We have,

$$\frac{\sin A}{c\sin B} + \frac{\sin B}{c} + \frac{\sin C}{b} = \frac{c}{ab} + \frac{b}{ac} + \frac{a}{bc}$$

$$\Rightarrow \frac{ka}{kbc} + \frac{b\sin B + c\sin C}{bc} = \frac{c^2+b^2}{abc} + \frac{a}{bc}$$

$$\Rightarrow \frac{b\sin B + c\sin C}{bc} = \frac{c^2+b^2}{abc}$$

$$\Rightarrow a = \frac{b^2+c^2}{b\sin B + c\sin C} \Rightarrow a = \frac{2bR\sin B + 2cR\sin C}{b\sin B + c\sin C}$$

$$\left[\because \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R\right]$$

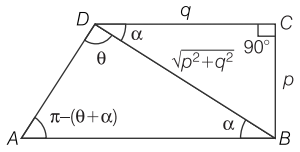
$\Rightarrow \quad a = 2R$

$\Rightarrow \quad 2R \sin A = 2R$

$\Rightarrow \quad \sin A = 1 \quad \Rightarrow \quad A = 90°$

9. ***(a)*** Let $\angle BDC = \alpha$. Then, $\tan \alpha = \frac{p}{q}$

Clearly, $\angle BDC = \angle ABD = \alpha$ $[\because AB \parallel CD]$



In ΔABD, using sine rule, we get

$$\frac{AB}{\sin\theta} = \frac{BD}{\sin\{\pi - (\theta + \alpha)\}} \Rightarrow \frac{AB}{\sin\theta} = \frac{\sqrt{p^2 + q^2}}{\sin(\theta + \alpha)}$$

$$\Rightarrow \quad AB = \frac{\sqrt{p^2 + q^2}\sin\theta}{\sin\theta\cos\alpha + \cos\theta\sin\alpha}$$

$$= \frac{\sqrt{p^2 + q^2}\sin\theta}{\sin\theta\left(\frac{q}{\sqrt{p^2 + q^2}}\right) + \cos\theta\left(\frac{p}{\sqrt{p^2 + q^2}}\right)}$$

$$= \frac{(p^2 + q^2)\sin\theta}{q\sin\theta + p\cos\theta}$$

10. ***(c)*** We have,

$$\frac{5}{4}\cos^2 2x + \cos^4 x + \sin^4 x + \cos^6 x + \sin^6 x = 2$$

$$\Rightarrow \frac{5}{4}\cos^2 2x + (\cos^2 x + \sin^2 x)^2 - 2\cos^2 x \sin^2 x + (\cos^2 x + \sin^2 x)^3 - 3\cos^2 x \sin^2 x (\cos^2 x + \sin^2 x) = 2$$

$$\Rightarrow \frac{5}{4}\cos^2 2x + 1 - 2\cos^2 x \sin^2 x + 1 - 3\cos^2 x \sin^2 x = 2$$

$$\Rightarrow \frac{5}{4}\cos^2 2x - \frac{1}{2}\sin^2 2x - \frac{3}{4}\sin^2 2x = 0$$

$$\Rightarrow \quad \frac{5}{4}\cos^2 2x - \frac{5}{4}\sin^2 2x = 0 \Rightarrow \quad \tan^2 2x = 1$$

$$\Rightarrow \quad \tan 2x = \pm 1, x \in [0, 2\pi]$$

$$\Rightarrow \quad 2x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}, \frac{11\pi}{4}, \frac{13\pi}{4}, \frac{15\pi}{4}$$

$$\Rightarrow \quad x = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}, \frac{13\pi}{8}, \frac{15\pi}{8}$$

Clearly, these are 8 solutions.

11. ***(d)*** We have, $\sin x + 2\sin 2x - \sin 3x = 3$

$\Rightarrow \quad \sin x + 4\sin x \cos x - 3\sin x + 4\sin^3 x = 3$

$\Rightarrow \quad -2\sin x + 4\sin x \cos x + 4\sin^3 x = 3$

$\Rightarrow \quad \sin x[-2 + 4\cos x + 4(1 - \cos^2 x)] = 3$

$\Rightarrow \quad \sin x[2 + 4\cos x - 4\cos^2 x] = 3$

$\Rightarrow \quad \sin x[3 - (4\cos^2 x - 4\cos x + 1)] = 3$

$\Rightarrow \quad \sin x[3 - (2\cos x - 1)^2] = 3$

$\Rightarrow \quad \sin x = 1, 3 - (2\cos x - 1)^2 = 3$

$\Rightarrow \quad \sin x = 1$ and $2\cos x - 1 = 0$

$\Rightarrow x = \frac{\pi}{2}$ and $x = \frac{\pi}{3}$, which is not possible at the same time.

Hence, the equation has no solution.

12. ***(c)*** We have,

$$(y + z)\cos 3\theta = xyz \sin 3\theta \quad ...(i)$$

$$x \sin 3\theta = \frac{2\cos 3\theta}{y} + \frac{2\sin 3\theta}{z} \quad ...(ii)$$

$$xyz \sin 3\theta = (y + 2z)\cos 3\theta + y \sin 3\theta \quad ...(iii)$$

From Eqs. (i) and (ii), we get

$$y(\cos 3\theta - 2\sin 3\theta) - z\cos 3\theta = 0 \quad ...(iv)$$

From Eqs. (ii) and (iii), we get

$$y(\cos 3\theta - \sin 3\theta) = 0 \quad ...(v)$$

Since, the given system of equations have a solution (x_0, y_0, z_0) such that $y_0 z_0 \neq 0$

$\therefore \quad \cos 3\theta - \sin 3\theta = 0$ [from Eq. (v)]

$\Rightarrow \quad \tan 3\theta = 1$

$\Rightarrow \quad 3\theta = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}$ $[\because 0 < \theta < \pi]$

$\Rightarrow \quad \theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{3\pi}{4}$

Hence, there are three values of θ.

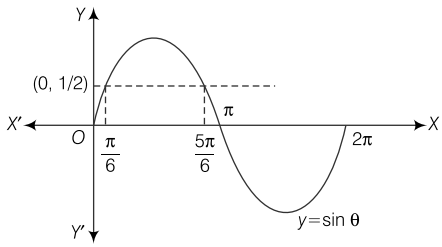
13. ***(a)*** We have,

$$2\sin^2\theta - 5\sin\theta + 2 > 0$$

$\Rightarrow \quad (2\sin\theta - 1)(\sin\theta - 2) > 0$

$\Rightarrow \quad 2\sin\theta - 1 < 0$ $[\because \sin\theta - 2 < 0]$

$\Rightarrow \quad \sin\theta < \frac{1}{2}$



$\Rightarrow \quad \theta \in \left(0, \frac{\pi}{6}\right) \cup \left(\frac{5\pi}{6}, 2\pi\right)$

14. ***(c, d)*** We have,

$$\sum_{m=1}^{6} \operatorname{cosec}\left(\theta + (m-1)\frac{\pi}{4}\right)\operatorname{cosec}\left(\theta + \frac{m\pi}{4}\right) = 4\sqrt{2}$$

$$\Rightarrow \quad \sum_{m=1}^{6} \frac{\sin\left\{\left(\theta + \frac{m\pi}{4}\right) - \left(\theta + (m-1)\frac{\pi}{4}\right)\right\}}{\sin\left(\theta + \frac{m\pi}{4}\right)\sin\left(\theta + (m-1)\frac{\pi}{4}\right)} = 4$$

$$\Rightarrow \quad \sum_{m=1}^{6}\left\{\cot\left(\theta + (m-1)\frac{\pi}{4}\right) - \cot\left(\theta + \frac{m\pi}{4}\right)\right\} = 4$$

$$\Rightarrow \quad \cot\theta - \cot\left(\frac{3\pi}{2} + \theta\right) = 4$$

$\Rightarrow \quad \cot\theta + \tan\theta = 4 \quad \Rightarrow \quad \tan^2\theta + 1 = 4\tan\theta$

$\Rightarrow \quad \tan^2\theta - 4\tan\theta + 1 = 0$

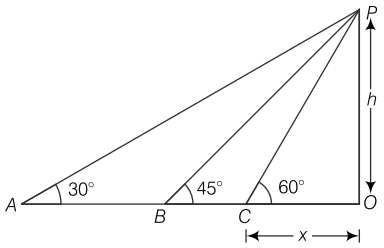
$\Rightarrow \quad \tan\theta = 2 \pm \sqrt{3}$

$\Rightarrow \quad \theta = \frac{\pi}{12}$ or $\frac{5\pi}{12}$

15. ***(c)*** Let OP be the tower of height h and let $OC = x$.

In Δ's COP, BOP and AOP, we have

$$\tan 60° = \frac{h}{x}, \tan 45° = \frac{h}{x + BC}$$



and $\tan 30° = \dfrac{h}{x + BC + AB}$

$\Rightarrow \quad h = \sqrt{3}x,\ h = x + BC$ and $\sqrt{3}\ h = x + BC + AB$

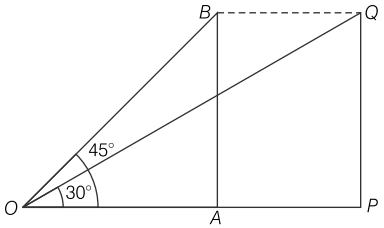
$\Rightarrow \quad \sqrt{3}x = x + BC$ and $\sqrt{3}\ (x + BC) = x + BC + AB$

$\Rightarrow (\sqrt{3} - 1)x = BC$ and $(\sqrt{3} - 1)x = BC\ (1 - \sqrt{3}) + AB$

$\Rightarrow \quad BC = BC\ (1 - \sqrt{3}) + AB$

$\Rightarrow \quad \sqrt{3}\ BC = AB \quad \Rightarrow\ AB : BC = \sqrt{3} : 1$

16. ***(b)*** Let AB be the vertical pole of height 20 m and let Q be the position of bird after one second. It is given the $\angle AOB = 45°$ and $\angle POQ = 30°$.



In ΔOAB and ΔOPQ, we have

$$\tan 45° = \frac{AB}{OA} \quad \text{and} \quad \tan 30° = \frac{PQ}{OP}$$

$\Rightarrow \quad OA = 20$ and $\sqrt{3}\ PQ = OP$

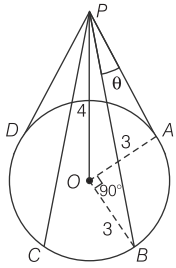
$\Rightarrow \quad OP = 20\sqrt{3} \qquad [\because PQ = OA = 20]$

$\Rightarrow \quad OA + AP = 20\sqrt{3}$

$\Rightarrow\ AP = 20\sqrt{3} - 20 = 20\ (\sqrt{3} - 1)$

$\therefore$ Speed of bird $= 20\ (\sqrt{3} - 1)$ m/s

17. ***(c)*** Let O be the centre of the circular ring which is suspended by the strings PA, PB, PC and PD in such a way that P is just above the point O and arc AB = arc BC = arc CD = arc AD.



$\therefore \quad \angle AOB = 90°$

Also, in ΔAOP, we have

$$AP = \sqrt{OA^2 + OP^2} = \sqrt{3^2 + 4^2} = 5$$

$\Rightarrow\ BP = AP = 5$

In ΔAPB, we have

$$\cos\theta = \frac{AP^2 + BP^2 - AB^2}{2AP \cdot BP} = \frac{5^2 + 5^2 - (3\sqrt{2})^2}{2 \times 5 \times 5}$$

$[\text{in } \Delta AOB,\ AB^2 = OA^2 + OB^2]$

$$\Rightarrow \quad \cos\theta = \frac{16}{25}$$

18. ***(b)*** We have, $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

$\Rightarrow \quad c^2 - (2b \cos A)\ c + (b^2 - a^2) = 0$

Since, c_1 and c_2 are the roots of this equation.

$\therefore\ c_1 + c_2 = 2b \cos A$ and $c_1 c_2 = b^2 - a^2$

Now, $(c_1 - c_2)^2 + (c_1 + c_2)^2 \tan^2 A$

$= (c_1 + c_2)^2 - 4c_1c_2 + (c_1 + c_2)^2 \tan^2 A$

$= (c_1 + c_2)^2 \sec^2 A - 4c_1c_2$

$= 4b^2 \cos^2 A \sec^2 A - 4(b^2 - a^2)$

$= 4b^2 - 4b^2 + 4a^2 = 4a^2$

19. ***(d)*** We have,

Area of ΔABC = Area of ΔABD + Area of ΔACD

$$\Rightarrow \frac{1}{2}\ bc \sin A = \frac{1}{2}\ c \times AD \times \sin\frac{A}{2} + \frac{1}{2}\ b \times AD \times \sin\frac{A}{2}$$

$$\Rightarrow \quad AD = \frac{2bc}{b + c} \cos\frac{A}{2}$$

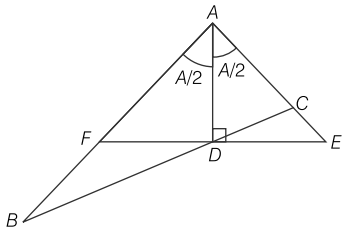
So, option (a) is correct.

In ΔAED, we have

$$\cos\frac{A}{2} = \frac{AD}{AE} \quad \Rightarrow \quad AE = AD \sec\frac{A}{2}$$

$$\Rightarrow \quad AE = \frac{2bc}{b + c} \Rightarrow AE \text{ is the HM of } b \text{ and } c.$$

So, option (b) is correct.



In ΔAEF, we have

$AD \perp EF$ and AD is the bisector of $\angle A$.

$\therefore \quad ED = DF$

$$\Rightarrow EF = 2DE = 2AD \tan\frac{A}{2} = 2 \times \frac{2bc}{b + c} \cos\frac{A}{2} \cdot \tan\frac{A}{2}$$

$$\therefore \quad EF = \frac{4bc}{b + c} \sin\frac{A}{2}$$

So, option (c) is correct.

20. ***(a)*** $\sin^2\theta + 3\sin\theta\cos\theta + 5\cos^2\theta$

$$= \frac{1}{2}\ [1 - \cos 2\theta + 3\sin 2\theta + 5\ (1 + \cos 2\theta)]$$

$$= 3 + 2\cos 2\theta + \frac{3}{2}\sin 2\theta$$

Clearly, $\quad -\sqrt{4 + \dfrac{9}{4}} \le 2\cos 2\theta + \dfrac{3}{2}\sin 2\theta \le \sqrt{4 + \dfrac{9}{4}}$

$$\Rightarrow \quad -\frac{5}{2} \le 2\cos 2\theta + \frac{3}{2}\sin 2\theta \le \frac{5}{2}$$

$$\Rightarrow \quad \frac{1}{2} \le 3 + 2\cos 2\theta + \frac{3}{2}\sin 2\theta \le \frac{11}{2}$$

$$\Rightarrow \quad \frac{2}{11} \le \frac{1}{3 + 2\cos 2\theta + \dfrac{3}{2}\sin 2\theta} \le 2$$

$$\Rightarrow \quad \frac{2}{11} \le \frac{1}{\sin^2\theta + 3\sin\theta\cos\theta + 5\cos^2\theta} \le 2$$

Hence, the maximum value of the given expression is 2.

APPLICATION OF DERIVATIVE

A collection of chapterwise Statement type problems for JEE Main

Directions (Q. Nos. 1-30) Each question has four choices a, b, c and d, out of which only one is correct. Each question contains Statement I and Statement II.

(a) If both the Statements are true and Statement II is the correct explanation of Statement I

(b) If both the Statements are true but Statement II is not the correct explanation of Statement I

(c) If Statement I is true and Statement II is false

(d) If Statement I is false and Statement II is true

1. **Statement I** The maximum value of $f(x) = \dfrac{1}{3x^4 + 8x^3 - 18x^2 + 60}$ is $\dfrac{1}{53}$.

Statement II The function $g(x) = \dfrac{1}{f(x)}$ attains its minimum value at $x = 1$ and $x = -3$.

2. Let $f(x) = \begin{cases} e^x + 1, & -1 \le x < 0 \\ e^x, & x = 0 \\ e^x - 1, & 0 < x \le 1 \end{cases}$

Statement I f is bounded but never attains its maximum and minimum values.

Statement II $x = 0$ is the point of discontinuity of $f(x)$.

3. **Statement I** The critical points of $f(x) = x \cos x$ occur in $\left(\dfrac{\pi}{4}, \dfrac{\pi}{3}\right)$.

Statement II The function $g(x) = x \tan x$ increases on $\left(0, \dfrac{\pi}{2}\right)$.

4. Let $f(x) = 2\sin x + \tan x - 3x$

Statement I $f(x)$ does not attain extreme values in $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$.

Statement II $f(x)$ is strictly increasing on $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$.

5. Let $f(x) = \tan^{-1}\left(\dfrac{1-x}{1+x}\right)$

Statement I The difference of the greatest and smallest values of $f(x)$ on $[0, 1]$ is $f(0) - f(1) = \dfrac{\pi}{4}$.

Statement II $g(x) = \tan^{-1} x$ is an increasing function on $[0, \infty)$.

6. Let $f : R \to R$ be a continuous function defined by

$$f(x) = \frac{1}{e^x + 2e^{-x}}$$

Statement I $f(c) = \dfrac{1}{3}$ for some $c \in R$.

Statement II $0 < f(x) \le \dfrac{1}{2\sqrt{2}}$, $\forall\, x \in R$.

7. Let f be a function defined by $f(x) = \begin{cases} \dfrac{\tan x}{x}, & x \ne 0 \\ 1, & x = 0 \end{cases}$

Statement I $x = 0$ is a point of minima.

Statement II $f'(0) = 0$

8. **Statement I** $e^\pi > \pi^e$

Statement II The function $x^{1/x}$ $(x > 0)$ is strictly decreasing in $[e, \infty)$.

9. Let $f(x) = \tan^{-1} x$ and $g(x) = x - \dfrac{x^3}{6}$.

Statement I $f(x) < g(x)$ for $0 < x \le 1$.

Statement II $h(x) = \tan^{-1} x - x + \dfrac{x^3}{6}$ decreases on $[-1, 1]$.

10. **Statement I** $e^x + e^{-x} > 2 + x^2, \forall x \neq 0$.

Statement II $f(x) = e^x + e^{-x} - 2 - x^2$ is an increasing function on R.

11. **Statement I** $\frac{2}{\pi} < \frac{\sin x}{x} < 1$ for $0 < x < \frac{\pi}{2}$.

Statement II $f(x) = \frac{\sin x}{x}$ is decreasing function on $\left(0, \frac{\pi}{2}\right]$.

12. Let $f(x) = 2\tan^{-1}\left(\frac{1-x}{1+x}\right)$ be defined on $[0, 1]$.

Statement I The range of $f(x)$ is $\left[0, \frac{\pi}{2}\right]$.

Statement II $f(x)$ decreases in $[0, 1]$.

13. Let $f(x) = \frac{20}{4x^3 - 9x^2 + 6x}$

Statement I Range of $f = [6, 20]$.

Statement II $f(x)$ increases on $\left(\frac{1}{2}, 1\right)$ and decreases on $(1, \infty) \cup (-\infty, 0) \cup (0, 1/2)$.

14. **Statement I** For $0 \le p < 1$ and for any positive a and b the inequality $(a+b)^p < a^p + b^p$ is valid.

Statement II For $0 \le p \le 1$, the function $f(x) = 1 + x^p - (1+x)^p$ decreases on $[0, \infty)$.

15. **Statement I** Let $f(x)$ and $g(x)$ be two real functions connected by the relation.

$$g(x) = f(x) - 2\{f(x)\}^2 + 4\{f(x)\}^3$$

Then, $f(x)$ and $g(x)$ increases or decreases together.

Statement II If $b^2 - 4ac < 0$ and $a > 0$, then

$$ax^2 + bx + c > 0, \forall x \in R.$$

16. **Statement I** The equation $3x^5 + 15x - 18 = 0$ has exactly one real root.

Statement II Between any two roots of a polynomial $f(x)$, there is a root of its derivative $f(x)$.

17. **Statement I** If f is differentiable on an open interval (a, b) such that $|f'(x)| \le M, \forall x \in (a, b)$, then

$$|f(x) - f(y)| \le M|x - y|, \forall x, y \in (a, b).$$

Statement II If $f(x)$ is a continuous function defined on $[a, b]$ such that it is differentiable on (a, b), then there exists $c \in (a, b)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.

18. **Statement I** There is no value of k for which the equation $x^3 - 3x + k = 0$ has two distinct roots between 0 and 1.

Statement II $x > \sin x, \forall x > 0$.

19. **Statement I** The equation $e^{x-1} + x - 2 = 0$ has only one real root.

Statement II Between any two roots of an equation $f(x) = 0$ there is a root of its derivative $f'(x) = 0$.

20. **Statement I** Lagrange's mean value theorem is not applicable to $f(x) = |x - 1|(x - 1)$.

Statement II $|x - 1|$ is not differentiable at $x = 1$.

21. **Statement I** If $27a + 9b + 3c + d = 0$, then the equation $f(x) = 4ax^3 + 3bx^2 + 2cx + d = 0$ has atleast one real root lying between $(0, 3)$.

Statement II If $f(x)$ is continuous in $[a, b]$, derivable in (a, b) such that $f(a) = f(b)$, then there exists atleast one point $c \in (a, b)$ such that $f'(c) = 0$.

22. **Statement I** If both functions $f(t)$ and $g(t)$ are continuous on the closed interval $[a, b]$, differentiable on the open interval (a, b) and $g'(t)$ is not zero on that open interval, then there exists some c in (a, b) such that $\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$.

Statement II If $f(t)$ and $g(t)$ are continuous and differentiable in $[a, b]$, then there exists some c in (a, b) such that $f'(c) = \frac{f(b) - f(a)}{b - a}$ and $g'(c) = \frac{g(b) - g(a)}{b - a}$ from Lagrange's mean value theorem.

23. **Statement I** The maximum value of $(\sqrt{-3 + 4x - x^2} + 4)^2 + (x - 5)^2$ (where $1 \le x \le 3$) is 36.

Statement II The maximum distance between the points $(5, -4)$ and the point on the circle $(x - 2)^2 + y^2 = 1$ is 6.

24. **Statement I** If $g(x)$ is a differentiable function, $g(2) \neq 0$, $g(-2) \neq 0$ and Rolle's theorem is not applicable to $f(x) = \frac{x^2 - 4}{g(x)}$ in $[-2, 2]$, then $g(x)$ has atleast one root in $(-2, 2)$.

Statement II If $f(a) = f(b)$, then Rolle's theorem is applicable for $x \in (a, b)$.

25. **Statement I** The tangent at $x = 1$ to the curve $y = x^3 - x^2 - x + 2$ again meets the curve at $x = 0$.

Statement II When the equation of a tangent is solved with the given curve, repeated roots are obtained at point of tangency.

26. Consider, a curve $C : y = \cos^{-1}(2x - 1)$ and a straight line $L : 2px - 4y + 2\pi - p = 0$.

Statement I The set of values of p for which the line L intersects the curve at three distinct points is $[-2\pi, -4)$.

Statement II The line L is always passing through point of inflection of the curve C.

27. **Statement I** If $f(x)$ is differentiable in $[0, 1]$ such that $f(0) = f(1) = 0$, then for any $\lambda \in R$, there exists c such that $f'(c) = \lambda f(c)$, $0 < c < 1$.

Statement II If $g(x)$ is differentiable in $[0, 1]$, where $g(0) = g(1)$, then there exists c such that $g'(c) = 0$, $0 < c < 1$.

28. **Statement I** For the function $f(x) = x^2 + 3x + 2$, LMVT is applicable in $[1, 2]$ and the value of c is $3/2$.

Statement II If LMVT is known to be applicable for any quadratic polynomial in $[a, b]$, then c of LMVT is $\frac{(a+b)}{2}$.

29. **Statement I** The value of $\left[\lim_{x \to 0^+} \frac{\sin x \tan x}{x^2}\right]$ is 1, where $[\cdot]$ denotes the greatest integer function.

Statement II For $\left(0, \frac{\pi}{2}\right)$, $\sin x < x < \tan x$.

30. **Statement I** The function $f(x) = x^4 - 8x^3 + 22x^2 - 24x + 21$ is decreasing for every $x \in (-\infty, 1) \cup (2, 3)$.

Statement II $f(x)$ is increasing for $x \in (1, 2) \cup (3, \infty)$ and has no point of inflection.

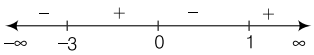
Answers with Explanation

1. ***(a)*** We have,

$$g(x) = \frac{1}{f(x)} = 3x^4 + 8x^3 - 18x^2 + 60$$

$$\Rightarrow \quad g'(x) = 12x^3 + 24x^2 - 36x = 12x(x+3)(x-1)$$

The signs of $g'(x)$ for different values of x are as shown below



So, $g(x)$ attains local minima at $x = -3$ and $x = 1$ values of $g(x)$ at these points are

$$g(-3) = -75 \text{ and } g(1) = 53$$

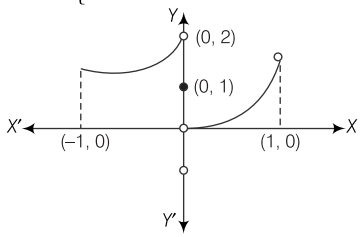
So, maximum value of $f(x) = \frac{1}{g(1)} = \frac{1}{53}$

Hence, both the Statements are true and Statement II is a correct explanation of Statement I.

2. ***(a)*** Given that,

$$f(x) = \begin{cases} e^x + 1, & -1 \le x < 0 \\ 1, & x = 0 \\ e^x - 1, & 0 < x \le 1 \end{cases}$$

$$\Rightarrow \quad f'(x) = \begin{cases} e^x, & -1 < x < 0 \\ 0, & x = 0 \\ e^x, & 0 < x < 1 \end{cases}$$



It is evident from the graph of $f(x)$, that $f(x)$ is bounded and has 2 and 0 as its maximum and minimum values. Also, $f(x)$ does not attain these values as $f(x)$ is not continuous at $x = 0$.

Hence, both the Statements are true and Statement II is a correct explanation of Statement I.

3. ***(b)*** It is given that $g(x) = x \tan x$

$$\therefore g'(x) = x \sec^2 x + \tan x > 0, \forall x \in \left(0, \frac{\pi}{2}\right)$$

$\Rightarrow g(x)$ is increasing on $\left(0, \frac{\pi}{2}\right)$.

So, Statement II is true.

Now, $\quad f(x) = x \cos x$

$$\Rightarrow \quad f'(x) = \cos x - x \sin x$$

$$\Rightarrow \quad f'(x) = \cos x(1 - x \tan x)$$

$$\Rightarrow \quad f'\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}\left(1 - \frac{\pi}{4}\right) \text{ and } f'\left(\frac{\pi}{3}\right) = \frac{1}{2}\left(1 - \frac{\pi}{\sqrt{3}}\right)$$

$$\Rightarrow \quad f'\left(\frac{\pi}{4}\right) > 0 \text{ and } f'\left(\frac{\pi}{3}\right) < 0$$

Also, $f'(x)$ is a continuous function.

So, $f'(x) = 0$ for some $c \in \left(\frac{\pi}{4}, \frac{\pi}{3}\right)$.

Hence, $f(x)$ has its critical points in $\left(\frac{\pi}{4}, \frac{\pi}{3}\right)$.

So, both the Statements are true but Statement II is not a correct explanation of Statement I.

4. ***(a)*** We have, $\quad f(x) = 2\sin x + \tan x - 3x$

$$\Rightarrow f'(x) = 2\cos x + \sec^2 x - 3$$

$$\Rightarrow f'(x) = \frac{2\cos^3 x + 1 - 3\cos^2 x}{\cos^2 x}$$

$$\Rightarrow f'(x) = \frac{(1 - \cos x)^2(1 + 2\cos x)}{\cos^2 x} > 0,$$

$$x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) - \{0\}$$

So, $f(x)$ is strictly increasing on $\left(\frac{-\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right)$.

Also, $f'(0) = 0$

Hence, $x = 0$ is the point of inflexion of $f(x)$.

Thus, both the Statements are true and Statement II is a correct explanation of Statement I.

5. ***(a)*** We have, $g(x) = \tan^{-1} x$

$$\Rightarrow \quad g'(x) = \frac{1}{1+x^2} > 0, \quad \forall\, x$$

$\Rightarrow g(x)$ is increasing on $[0, \infty)$.
So, Statement II is true.

Now, $\quad f(x) = \tan^{-1}\left(\frac{1-x}{1+x}\right)$

$$\Rightarrow \quad f(x) = \tan^{-1}(1) - \tan^{-1} x = \frac{\pi}{4} - g(x)$$

$\because\ g(x)$ is increasing on $[0, \infty)$.
$\Rightarrow -\, g(x)$ is decreasing on $[0, \infty)$.
$\Rightarrow \frac{\pi}{4} - g(x)$ is decreasing on $[0, \infty)$.
$\Rightarrow f(x)$ is decreasing on $[0, 1]$.
$\Rightarrow f(0) = \frac{\pi}{4} - 0 = \frac{\pi}{4}$ is the greatest value and

$f(1) = \frac{\pi}{4} - g(1) = \frac{\pi}{4} - \frac{\pi}{4} = 0$ is the least value of $f(x)$.

Hence, required difference $= \frac{\pi}{4} - 0 = \frac{\pi}{4}$

Thus, both the Statements are true and Statement II is a correct explanation of Statement I.

6. ***(a)*** We have, $\quad f(x) = \frac{1}{e^x + 2e^{-x}} \quad \Rightarrow \quad f(x) = \frac{e^x}{e^{2x} + 2}$

$$\Rightarrow \quad f'(x) = \frac{(e^{2x}+2)e^x - e^x(2e^{2x})}{(e^{2x}+2)^2} = \frac{e^x(2 - e^{2x})}{(e^{2x}+2)^2}$$

For $f(x)$ to attain a local maximum or minimum, we must have

$$f'(x) = 0 \Rightarrow e^{2x} = 2 \Rightarrow e^x = \sqrt{2}$$

For this value of e^x, we have $f(x) = \frac{1}{2\sqrt{2}}$

Clearly, $\quad f(x) > 0$

$$\therefore \quad 0 < f(x) \le \frac{1}{2\sqrt{2}}, \quad \forall\, x \in R$$

So, Statement II is true.
Since, $f(x)$ is a continuous function and every continuous function attains every value between its maximum and minimum.
As, $0 < \frac{1}{3} < \frac{1}{2\sqrt{2}}$. So, there exists $c \in R$ such that $f(c) = \frac{1}{3}$.
Hence, both the Statements are true and Statement II is a correct explanation of Statement I.

7. ***(b)*** We observed that

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{\frac{\tan x}{x} - 1}{x}$$

$$= \lim_{x \to 0} \frac{\tan x - x}{x^2} = \lim_{x \to 0} \frac{\sec^2 x - 1}{2x} = \frac{1}{2} \lim_{x \to 0} \frac{\tan^2 x}{x} = 0$$

So, Statement II is correct.

Now, $f(x) = \frac{\tan x}{x} \Rightarrow f'(x) = \frac{x \sec^2 x - \tan x}{x^2} = \frac{2x - \sin 2x}{2x^3 \cos^2 x}$

As, $\sin x < x$ for $x > 0$ and $x < \sin x$ for $x < 0$. Therefore, $f'(x)$ changes its sign from negative to positive as x passes through 0. Hence, $x = 0$ is a point of local minimum of $f(x)$.
Thus, Statement II is not a correct explanation of Statement I.

8. ***(a)*** Let $\quad f(x) = x^{1/x}$. Then,

$$f'(x) = x^{1/x}\left[\frac{1}{x^2} - \frac{\log x}{x^2}\right] = x^{1/x}\left[\frac{1 - \log x}{x^2}\right]$$

$$\Rightarrow \quad f'(x) < 0, \ \forall\, x \in (e, \infty)$$

$\Rightarrow f(x)$ is strictly decreasing in $[e, \infty)$.

$$\Rightarrow \quad f(e) > f(\pi) \qquad [\because e < \pi]$$

$$\Rightarrow \quad e^{1/e} > \pi^{1/\pi} \ \Rightarrow e^\pi > \pi^e$$

Hence, both the Statements are true and Statement II is a correct explanation of Statement I.

9. ***(a)*** We have, $h(x) = \tan^{-1} x - x + \frac{x^3}{6}$

$$\Rightarrow \quad h'(x) = \frac{1}{1+x^2} - 1 + \frac{x^2}{2} = \frac{x^2(x-1)(x+1)}{2(1+x^2)}$$

$\Rightarrow h'(x) < 0, \ \forall\, x \in (-1, 1)$
$\Rightarrow h(x)$ decreases on $[-1, 1]$.
$\Rightarrow h(x) < h(0), \ \forall\, x \in (0, 1]$
$\Rightarrow f(x) - g(x) < 0, \ \forall\, x \in (0, 1] \qquad [\because h(x) = f(x) - g(x)]$
$\Rightarrow f(x) < g(x), \ \forall\, x \in (0, 1]$
Hence, both the Statements are true and Statement II is a correct explanation of Statement I.

10. ***(a)*** We have,

$$f(x) = e^x + e^{-x} - 2 - x^2$$

$$\Rightarrow \quad f'(x) = e^x - e^{-x} - 2x$$

$$\Rightarrow \quad f''(x) = e^x + e^{-x} - 2 = (e^x - e^{-x})^2 > 0, \ \forall\, x \neq 0$$

$\Rightarrow f'(x)$ is increasing in R.
$\Rightarrow f'(x) > f'(0), \ \forall\, x \in R, x \neq 0 \Rightarrow f'(x) > 0, \ \forall\, x \in R - \{0\}$
$\Rightarrow f(x)$ is increasing in R.
$\Rightarrow f(x) > f(0), \ \forall\, x \neq 0 \Rightarrow e^x + e^{-x} - 2 - x^2 > 0, \ \forall\, x \neq 0$
$\Rightarrow e^x + e^{-x} > 2 + x^2, \ \forall\, x \neq 0$

Hence, both the Statements are true and Statement II is a correct explanation of Statement I.

11. ***(a)*** We have, $f(x) = \frac{\sin x}{x} \Rightarrow f'(x) = \frac{x \cos x - \sin x}{x^2}$

$$\Rightarrow \quad f'(x) = \frac{g(x)}{x^2}, \text{ where } g(x) = x \cos x - \sin x$$

Now, $\quad g(x) = x \cos x - \sin x$

$$\Rightarrow \quad g'(x) = -x \sin x < 0, \ \forall\, x \in \left(0, \frac{\pi}{2}\right]$$

$\Rightarrow g(x)$ is strictly decreasing in $\left[0, \frac{\pi}{2}\right]$.

$\Rightarrow g(x) < g(0), \ \forall\, x \in \left(0, \frac{\pi}{2}\right] \ \Rightarrow g(x) < 0, \ \forall\, x \in \left(0, \frac{\pi}{2}\right]$

$\Rightarrow f'(x) = \frac{g(x)}{x^2} < 0, \ \forall\, x \in \left(0, \frac{\pi}{2}\right]$

$\Rightarrow f(x)$ is decreasing in $x \in \left(0, \frac{\pi}{2}\right]$.

$\Rightarrow f\left(\frac{\pi}{2}\right) < f(x) < \lim_{x \to 0} f(x), \ \forall\, x \in \left(0, \frac{\pi}{2}\right]$

$\Rightarrow \frac{2}{\pi} < \frac{\sin x}{x} < 1, \ \forall\, x \in \left(0, \frac{\pi}{2}\right]$

Hence, both the Statements are true and Statement II is a correct explanation of Statement I.

12. ***(a)*** We have, $f(x) = 2\tan^{-1}\left(\frac{1-x}{1+x}\right)$

$= 2(\tan^{-1}1 - \tan^{-1}x) = \frac{\pi}{2} - 2\tan^{-1}x$

$\Rightarrow \quad f'(x) = -\frac{2}{1+x^2} < 0, \forall\ x \in [0, 1]$

$\Rightarrow f(x)$ decreases on $[0, 1]$.

$\Rightarrow$ Range of $f(x) = [f(1), f(0)] = \left[0, \frac{\pi}{2}\right]$

Hence, both the Statements are true and Statement II is a correct explanation of Statement I.

13. ***(d)*** We have, $f(x) = \frac{20}{x\left\{\left(2x - \frac{9}{4}\right)^2 + \frac{15}{16}\right\}}$

$\Rightarrow \quad \lim_{x \to 0^-} f(x) = -\infty$ and $\lim_{x \to 0^+} f(x) = \infty$

Also, $f(x)$ is continuous on $R - \{0\}$.

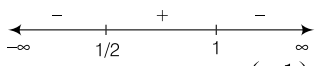
So, range of $f(x) = (-\infty, \infty)$.

Hence, Statement I is false.

Again, $f(x) = \frac{20}{4x^3 - 9x^2 + 6x}$

$\Rightarrow \quad f'(x) = \frac{-20(12x^2 - 18x + 6)}{(4x^3 - 9x^2 + 6x)^2} = \frac{-120(2x-1)(x-1)}{(4x^3 - 9x^2 + 6x)^2}$

The signs of $f'(x)$ for different values of x are as shown below



So, $f(x)$ is decreasing on $(-\infty, 0) \cup \left(0, \frac{1}{2}\right) \cup (1, \infty)$ and increasing on $\left(\frac{1}{2}, 1\right)$.

Hence, Statement II is true.

14. ***(c)*** We have, $f(x) = 1 + x^p - (1+x)^p$

$\Rightarrow \quad f'(x) = px^{p-1} - p(1+x)^{p-1}$

$= p\left\{\frac{1}{x^{1-p}} - \frac{1}{(1+x)^{1-p}}\right\}$

Now, $1 + x > x$ for $x > 0$

$\Rightarrow \quad \frac{1}{1+x} < \frac{1}{x} \Rightarrow \frac{1}{(1+x)^{1-p}} < \frac{1}{x^{1-p}}$

$\Rightarrow \quad f'(x) > 0$ for $x > 0$

$\Rightarrow f(x)$ is increasing on $(0, \infty)$.

So, Statement II is not correct.

Now, $f(x)$ is increasing for $x > 0$

$\Rightarrow \quad f(x) > f(0), \ \forall\ x > 0$

$\Rightarrow \quad 1 + x^p > (1+x)^p, \forall\ x > 0$

$\Rightarrow \quad 1 + \left(\frac{a}{b}\right)^p > \left(1 + \frac{a}{b}\right)^p, \forall\ a > 0, b > 0$

$\Rightarrow \quad a^p + b^p > (a+b)^p, \forall\ a > 0, b > 0$

$\Rightarrow \quad (a+b)^p < a^p + b^p, \forall\ a > 0, b > 0$

Hence, Statement I is true.

15. ***(a)*** Clearly, Statement II is true.

Also, $g(x) = f(x) - 2(f(x))^2 + 4(f(x))^3$

$\Rightarrow \quad g'(x) = \{1 - 4f(x) + 12(f(x))^2\}f'(x) \quad \ldots(i)$

Using Statement II, we have

$12(f(x))^2 - 4f(x) + 1 > 0, \forall\ x$

$\Rightarrow g'(x)$ and $f'(x)$ are of the same sign.

$\Rightarrow g(x)$ and $f(x)$ increase and decrease together. [from Eq. (i)]

Hence, both the Statements are true and Statement II is a correct explanation of Statement I.

16. ***(a)*** Statement II, being algebraic interpretation of Rolle's theorem, is true.

Let x_1, x_2 be two roots of $f(x) = 3x^5 + 15x - 18$.

Then, $f'(x) = 12x^4 + 15$ must have a root between x_1 and x_2. But, $f'(x) > 0, \forall\ x$.

So, it has no real root. This contradicts the algebraic interpretation of Rolle's theorem.

Hence, $f(x) = 3x^5 + 15x - 18$ has exactly one real root $x = 1$.

Thus, both the Statements are true and Statement II is a correct explanation of Statement I.

17. ***(a)*** Clearly, Statement II is true (Statement of Lagrange's mean value theorem).

Let $x, y \in (a, b)$ such that $x < y$.

Applying Lagrange's mean value theorem on $[x, y]$, we get

$f'(c) = \frac{f(x) - f(y)}{x - y}$ for some $c \in [x, y]$

$\Rightarrow \quad |f'(c)| = \frac{|f(x) - f(y)|}{|x - y|} \Rightarrow \frac{|f(x) - f(y)|}{|x - y|} \le M$

$\Rightarrow \quad |f(x) - f(y)| \le M\,|x - y|$

So, Statement I is true. Also, Statement II is a correct explanation of Statement I.

18. ***(b)*** Let there be two distinct roots of $f(x) = x^3 - 3x + k = 0$ in the interval $[0, 1]$. Then, $f'(x) = 0$, i.e. $3x^2 - 3 = 0$ must have a root between 0 and 1. But $3x^2 - 3 = 0$ does not have a root between 0 and 1. This contradicts the algebraic interpretation of Rolle's theorem. So, $f(x) = 0$ does not have two distinct roots between 0 and 1.

Hence, Statement I is true.

Let $g(x) = x - \sin x$ and $x > 0$ be any real number.

Applying Lagrange's mean value theorem on $[0, x]$, we have

$g'(c) = \frac{g(x) - g(0)}{x - 0}$ for some $c \in (0, x)$

$\Rightarrow \quad 1 - \cos c = \frac{x - \sin x}{x} \qquad [\because 1 - \cos c > 0]$

$\Rightarrow \quad \frac{x - \sin x}{x} > 0 \ \Rightarrow x - \sin x > 0 \qquad [\because x > 0]$

$\Rightarrow \quad x > \sin x$

So, Statement II is also true. But, it is not a correct explanation of Statement I.

19. ***(a)*** Clearly, Statement II is true.

Let $f(x) = e^{x-1} + x - 2$

We have, $f(1) = 0$. So, $x = 1$ is a root of $f(x) = 0$.

Let $x = \alpha$ be a root, other than 1 of $f(x) = 0$.

Then, $f'(x) = 0$ must have a real root between 1 and α.

Clearly, $f'(x) = 0$ does not have a real root. This contradicts the algebraic interpretation of Rolle's theorem.

So, $f(x) = 0$ has no real root other than 1.

Hence, Statement I is true and Statement II is a correct explanation of Statement I.

20. ***(d)*** Though $|x-1|$ is non-differentiable at $x=1$, but $(x-1)|x-1|$ is differentiable at $x=1$, for which Lagrange's mean value theorem is applicable.

21. ***(a)*** Consider, $f'(x) = 4ax^3 + 3bx^2 + 2cx + d$

$\therefore \quad f(x) = ax^4 + bx^3 + cx^2 + dx + e$

$f(0) = e$

and $\quad f(3) = 81a + 27b + 9c + 3d + e$

$= 3(27a + 9b + 3c + d) + e = e$

Hence, Rolle's theorem is applicable for $f(x)$.

Thus, there exists atleast one c in (a, b) such that $f'(c) = 0$.

22. ***(c)*** Statement I is correct as it is the Statement of Cauchy's mean value theorem. Statement II is false as it is necessary that c in both

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

and $\quad g'(c) = \frac{g(b)-g(a)}{b-a}$ be the same.

23. ***(a)*** Let $y = \sqrt{-3+4x-x^2}$ or $x^2 + y^2 - 4x + 3 = 0$

or point (x, y) lies on this circle.

Then, the given expression is $(y+4)^2 + (x-5)^2$, which is the square of distance between point $P(5, -4)$ and any point on the circle $x^2 + y^2 - 4x + 3 = 0$ which has centre $C(2, 0)$ and radius 1. Now, $CP = 5$. Then, the maximum distance between the point P and any point on the circle is 6.

Therefore, maximum value of

$$(\sqrt{-3+4x-x^2}+4)^2 + (x-5)^2 \text{ is } 36.$$

24. ***(c)*** Statement I is correct as $f(-2) = f(2) = 0$ and Rolle's theorem is not applicable. Then, it implies that either $f(x)$ is discontinuous or $f'(x)$ does not exist at atleast one point in $(-1, 1)$. Since, it is given that $g(x)$ is differentiable, $g(x) = 0$ has atleast one value of x in $(-1, 1)$. Statement II is false as $f(x)$ must be differentiable in (a, b) is not given.

25. ***(d)*** When $x = 1$, $y = 1$

$$\frac{dy}{dx} = 3x^2 - 2x - 1 \text{ or } \left(\frac{dy}{dx}\right)_{x=1} = 0$$

Therefore, equation of the tangent is $y = 1$.

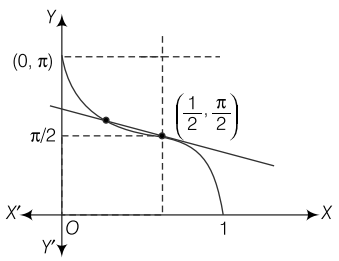
Solving with the curve, $x^3 - x^2 - x + 2 = 1$

or $x^3 - x^2 - x + 1 = 0$ or $x = -1, 1$ (1 is repeated root).

Therefore, the tangent meets the curve again at $x = -1$.

Hence, Statement I is false and Statement II is true.

26. ***(b)*** Point of inflection of the curve is $\left(\frac{1}{2}, \frac{\pi}{2}\right)$ and this satisfies the line L.



Slope of the tangent to the curve C at $\left(\frac{1}{2}, \frac{\pi}{2}\right)$,

$$\frac{dy}{dx} = \frac{-2}{\sqrt{1-(2x-1)^2}} = \frac{-1}{\sqrt{x-x^2}} = -(x-x^2)^{-1/2}$$

$$\frac{d^2y}{dx^2} = \frac{1}{2}\cdot\frac{(1-2x)}{(x-x^2)^{3/2}} = 0; x = \frac{1}{2}$$

$$\left.\frac{dy}{dx}\right|_{x=\frac{1}{2}} = -2$$

As the slope decreases from -2, line cuts the curve at three distinct points and the minimum slope of the line when it intersects the curve at three distinct points is

$$\frac{\pi - \frac{\pi}{2}}{0 - \frac{1}{2}} = -\pi$$

$\therefore \quad \frac{p}{2} \in [-\pi, -2)$ or $p \in [-2\pi, -4)$

27. ***(a)*** Consider,

$F(x) = e^{-\lambda x} f(x), \lambda \in R$

$F(0) = f(0) = 0$

$F(1) = e^{-1} f(1) = 0$

Therefore, by Rolle's theorem, $F'(c) = 0$

$F'(x) = e^{-\lambda x}\{f'(x) - \lambda f(x)\}$

$F'(c) = 0 \Rightarrow e^{-\lambda c}\{f'(c) - \lambda f(c)\} = 0$

or $\quad f'(c) = \lambda f(c), 0 < c < 1$

28. ***(a)*** Verify by taking $f(x) = lx^2 + mx + n$ in $[a, b]$.

29. ***(b)*** Let $f(x) = \sin x \tan x - x^2$

or $\quad f'(x) = \sin x \sec^2 x + \sin x - 2x$

or $\quad f''(x) = 2\sin x \sec^2 x \tan x + \sec x + \cos x - 2$

$= 2\sin x \tan x \sec^2 x + (\cos x + \sec x - 2)$

$> 0, \forall x \in \left(0, \frac{\pi}{2}\right)$

Thus, $f'(x)$ is an increasing function.

So, $\quad f'(x) > f'(0) \Rightarrow \sin x \sec^2 x + \sin x - 2x > 0$

Thus, $f(x)$ is an increasing function.

So, $\quad f(x) > f(0) \Rightarrow \sin x \tan x - x^2 > 0$

$\Rightarrow \quad \sin x \tan x > x^2$

Thus, Statement I is true. Also, Statement II is true but it does not explain Statement I.

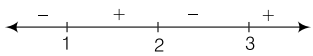
30. ***(a)*** $f(x) = x^4 - 8x^3 + 22x^2 - 24x + 21$

$\Rightarrow f'(x) = 4x^3 - 24x^2 + 44x - 24$

$= 4(x^3 - 6x^2 + 11x - 6)$

$= 4(x-1)(x-2)(x-3)$

Sign scheme of $f'(x)$



From the sign scheme of $f'(x)$, $f(x)$ increases for $x \in (1, 2) \cup (3, \infty)$.

Since, $f(x)$ is a polynomial function, which is continuous and has no point of inflection, intervals of increase and decrease occur alternatively.

CONCEPT MAP your Revision Tool

3-DIMENSIONAL COORDINATE GEOMETRY

① PLANE

A plane is a surface such that if any two points are taken on it, the line segment joining them lies completely on the surface. The general equation of the plane is $ax + by + cz + d = 0$.

② VECTOR EQUATION OF A PLANE

The vector equation of a plane passing through a point having position vector $\mathbf{a}$ and normal to vector $\mathbf{n}$ is $(\mathbf{r} - \mathbf{a}) \cdot \mathbf{n} = 0 \Rightarrow \mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$.

③ ANGLE BETWEEN TWO PLANES

The angle θ between the planes $\mathbf{r} \cdot \mathbf{n}_1 = d_1$ and $\mathbf{r} \cdot \mathbf{n}_2 = d_2$ is given by $\cos\theta = \dfrac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|}$

(i) If the planes are perpendicular, then $\mathbf{n}_1 \cdot \mathbf{n}_2 = 0$

(ii) If the planes are parallel, then there exists a scalar λ such that $\mathbf{n}_1 = \lambda \mathbf{n}_2$

(iii) If $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are cartesian equations of two planes, then

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

(a) If the planes are perpendicular.

$\therefore \quad a_1a_2 + b_1b_2 + c_1c_2 = 0$

(b) If the planes are parallel.

$$\therefore \quad \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

④ EQUATION OF A PLANE IN PARAMETRIC FORM

The equation of the plane passing through a point having position vector $\mathbf{a}$ and parallel to $\mathbf{b}$ and $\mathbf{c}$ is $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$, where λ and μ are scalar.

⑤ PLANE PARALLEL TO A GIVEN PLANE

(i) **Vector form** Since, parallel planes have the common normal. Therefore, equation of a plane parallel to the plane $\mathbf{r} \cdot \mathbf{n} = d_1$ is $\mathbf{r} \cdot \mathbf{n} = d_2$, where d_2 is a constant determined by the given condition.

(ii) **Cartesian form** Let $ax + by + cz + d = 0$ be a plane. Then, direction ratios of its normal are a, b, c. Then, the equation of a plane parallel to the given plane is $ax + by + cz + k = 0$, where k is an arbitrary constant and is determined by the given condition.

⑥ SYMMETRICAL FORM OF EQUATION OF A LINE

The equation of a straight line passing through a fixed point $A(x_1, y_1, z_1)$ and having direction cosines l, m, n is $\dfrac{x - x_1}{l} = \dfrac{y - y_1}{m} = \dfrac{z - z_1}{n}$

(i) If a line passes through two points (x_1, y_1, z_1) and (x_2, y_2, z_2), then its direction ratios are proportional to $x_2 - x_1$, $y_2 - y_1$ and $z_2 - z_1$. So, the equation of line is

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

(ii) The vector equation of a straight line passing through a fixed point with a position vector $\mathbf{a}$ and parallel to $\mathbf{b}$ is $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$, where λ is scalar.

(iii) The vector equation of a line passing through two points with position vectors $\mathbf{a}$ and $\mathbf{b}$ is $\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a})$

⑦ ANGLE BETWEEN TWO LINES

Let the vector equations of the two lines be $\mathbf{r} = \mathbf{a}_1 + \lambda\mathbf{b}_1$ and $\mathbf{r} = \mathbf{a}_2 + \mu\mathbf{b}_2$. It θ is the angle between the given lines, then $\cos\theta = \dfrac{\mathbf{b}_1 \cdot \mathbf{b}_2}{|\mathbf{b}_1||\mathbf{b}_2|}$

⑧ PLANE THROUGH THE INTERSECTION OF TWO PLANES

(i) **Vector form** The equation of a plane passing through the intersection of the planes $\mathbf{r} \cdot \mathbf{n}_1 = d_1$ and $\mathbf{r} \cdot \mathbf{n}_2 = d_2$ is $\mathbf{r} \cdot (\mathbf{n}_1 + \lambda\mathbf{n}_2) = d_1 + \lambda d_2$, where λ is an arbitrary constant.

(ii) **Cartesian form** The equation of a plane passing through the intersection of $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ is

$$(a_1x + b_1y + c_1z + d_1) + \lambda(a_2x + b_2y + c_2z + d_2) = 0$$

where λ is a constant.

⑨ DISTANCE OF A POINT FROM A PLANE

(i) **Vector form** The length of the perpendicular from a point having position vector $\mathbf{a}$ to the plane $\mathbf{r} \cdot \mathbf{n} = d$ is given by $p = \dfrac{|\mathbf{a} \cdot \mathbf{n} - d|}{|\mathbf{n}|}$.

DIRECTION COSINES AND DIRECTION RATIOS

DIRECTION COSINES

If α, β, γ are the angles which a vector $\mathbf{OP}$ makes with the positive directions of the coordinate axes OX, OY, OZ respectively, then $\cos\alpha, \cos\beta, \cos\gamma$ are known as the direction cosines of $\mathbf{OP}$ and generally denoted by the letters l, m, n, respectively

$$l = \cos\alpha, m = \cos\beta, n = \cos\gamma$$

Let $P(x, y, z)$ be a point in space such that $\mathbf{r} = \mathbf{OP}$ has direction cosines l, m, n. Then,

(i) $l|\mathbf{r}|, m|\mathbf{r}|, n|\mathbf{r}|$ are projections of $\mathbf{r}$ on OX, OY, OZ respectively.

(ii) $x = l|\mathbf{r}|, y = m|\mathbf{r}|, z = n|\mathbf{r}|$

(iii) $\mathbf{r} = |\mathbf{r}|(l\hat{\mathbf{i}} + m\hat{\mathbf{j}} + n\hat{\mathbf{k}})$

(iv) $l^2 + m^2 + n^2 = 1$

DIRECTION RATIOS

Let l, m, n be direction cosines of a vector $\mathbf{r}$ and a, b, c be three numbers such that $\dfrac{l}{a} = \dfrac{m}{b} = \dfrac{n}{c}$.

(ii) **Cartesian form** The length of the perpendicular from a point $P(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$ is given by $\dfrac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$.

⑩ DISTANCE BETWEEN PARALLEL PLANES

Distance between parallel planes $ax + by + cz + d_1 = 0$ and $ax + by + cz + d_2 = 0$ is given by $\dfrac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}$

If $\mathbf{r} \cdot \mathbf{n} = d_1$ and $\mathbf{r} \cdot \mathbf{n} = d_2$ are vector equations in two parallel planes, then distance between them is given by $\dfrac{|d_1 - d_2|}{\mathbf{n}}$

⑪ PLANES BISECTING THE ANGLES BETWEEN TWO GIVEN PLANES

The equation of the planes bisecting the angles between the planes

$a_1x + b_1y + c_1z + d_1 = 0$ and

$a_2x + b_2y + c_2z + d_2 = 0$ are

$$\frac{(a_1x + b_1y + c_1z + d_1)}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \pm\frac{a_2x + b_2y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

⑫ DISTANCE OF A POINT FROM A LINE

Let $P(\alpha, \beta, \gamma)$ be the given point, and let $\dfrac{x - x_1}{a} = \dfrac{y - y_1}{b} = \dfrac{z - z_1}{c}$ be the given line. We may

Then, we say that the direction ratios of $\mathbf{r}$ are proportional to a, b, c.

DC's IN TERMS OF DR's

$$l = \pm \frac{a}{\sqrt{a^2+b^2+c^2}}, m = \pm \frac{b}{\sqrt{a^2+b^2+c^2}},$$

$$n = \pm \frac{c}{\sqrt{a^2+b^2+c^2}}$$

ANGLE BETWEEN TWO LINES

Let $\mathbf{a}$ and $\mathbf{b}$ be two vectors with DC's l_1, m_1, n_1 and l_2, m_2, n_2, respectively. Then, angle between these two vectors is given by

$$\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$$

or $$\cos\theta = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

If lines are perpendicular, then

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$$

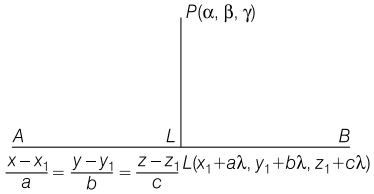
or $$a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$$

If lines are parallel, then

$$l_1 = l_2, m_1 = m_2, n_1 = n_2 \text{ or } \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}.$$

use the following steps to find the distance of P form the given line.

Step I Write the coordinates of a general point on the given line. The coordinates of general point on the line are $(x_1 + a\lambda, y_1 + b\lambda, z_1 + c\lambda)$, where λ is a parameter. Assume that this point L is the foot of the perpendicular drawn from P on the given line.



Step II Write direction ratios of PL.

Step III Apply the condition of perpendicularity of the given line and PL.

Step IV Obtain the value of λ from step III.

Step V Substitute λ in $(x_1 + a\lambda, y_1 + b\lambda, z_1 + c\lambda)$ to obtain the coordinates of L.

Step VI Obtain PL by using distance formula.

(13) IMAGE OF A POINT IN A PLANE

The coordinates of the image (α, β, γ) of a point (x_1, y_1, z_1) in the plane $ax + by + cz + d = 0$ are given by

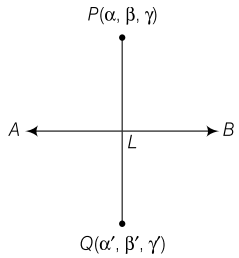
$$\frac{\alpha - x_1}{a} = \frac{\beta - y_1}{b} = \frac{\gamma - z_1}{c} = \frac{-2(ax_1 + by_1 + cz_1 + d)}{a^2+b^2+c^2}$$

The coordinates (α, β, γ) of the foot of the perpendicular from (x_1, y_1, z_1) on the plane $ax + by + cz + d = 0$ are given by

$$\frac{\alpha - x_1}{a} = \frac{\beta - y_1}{b} = \frac{\gamma - z_1}{c} = -\frac{(ax_1 + by_1 + cz_1)}{a^2+b^2+c^2}$$

(14) REFLECTION OR IMAGE OF A POINT IN A STRAIGHT LINE

Let $P(\alpha, \beta, \gamma)$ be the given point and let $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$ be the given line.



Let L be the foot of the perpendicular from P on the given line and let Q be the image of P in the given line. Let the coordinates of L be $(x_1 + a\lambda, y_1 + b\lambda, z_1 + c\lambda)$.

Then, direction ratios of PL are

$$x_1 + a\lambda - \alpha, y_1 + b\lambda - \beta, z_1 + c\lambda - \gamma$$

Since, PL is perpendicular to the given line, whose direction ratios are a, b, c. Therefore,

$$(x_1 + a\lambda - \alpha)a + (y_1 + b\lambda - \beta)b + (z_1 + c\lambda - \gamma)c = 0$$

$$\Rightarrow \lambda = -\frac{[a(x_1 - \alpha) + b(y_1 - \beta) + c(z_1 - \gamma)]}{a^2+b^2+c^2}$$

Substituting the value of λ in $(x_1 + a\lambda, y_1 + b\lambda, z_1 + c\lambda)$, we obtain coordinates of L.

Let the coordinates of Q be $(\alpha', \beta', \gamma')$. Since, l is the mid-point of PQ.

$$\therefore \quad \frac{\alpha + \alpha'}{2} = x_1 + a\lambda,$$

$$\frac{\beta + \beta'}{2} = y_1 + b\lambda$$

$$\frac{\gamma + \gamma'}{2} = z_1 + c\lambda$$

$$\Rightarrow \quad \alpha' = 2(x_1 + a\lambda) - \alpha, \beta' = 2(y_1 + b\lambda) - \beta,$$
$$\gamma' = 2(z_1 + c\lambda) - \gamma.$$

(15) SHORTEST DISTANCE BETWEEN TWO STRAIGHT LINES

(i) The shortest distance between the lines $\mathbf{r} = \mathbf{a}_1 + \lambda\mathbf{b}_1$ and $\mathbf{r} = \mathbf{a}_2 + \mu\mathbf{b}_2$ is given by

$$\left|\frac{(\mathbf{b}_1 \times \mathbf{b}_2)\cdot(\mathbf{a}_2 - \mathbf{a}_1)}{|\mathbf{b}_1 \times \mathbf{b}_2|}\right|$$

If the lines intersect, then

$$\left|\frac{(\mathbf{b}_1 \times \mathbf{b}_2)\cdot(\mathbf{a}_2 - \mathbf{a}_1)}{|\mathbf{b}_1 \times \mathbf{b}_2|}\right| = 0$$

$$\Rightarrow \quad \mathbf{b}_1 \times \mathbf{b}_2 \cdot (\mathbf{a}_2 - \mathbf{a}_1) = 0$$

$$\Rightarrow \quad [\mathbf{b}_1 \mathbf{b}_2 (\mathbf{a}_2 - \mathbf{a}_1)] = 0$$

(ii) If $\mathbf{r}_1 = \mathbf{a}_1 + \lambda\mathbf{b}$ and $\mathbf{r}_2 = \mathbf{a}_2 + \mu\mathbf{b}$, then shortest distance is $\left|\frac{(\mathbf{a}_2 - \mathbf{a}_1) \times \mathbf{b})}{|\mathbf{b}|}\right|$

(16) ANGLE BETWEEN A LINE AND A PLANE

(i) **Vector form** If θ is the angle between a line $\mathbf{r} = (\mathbf{a} + \lambda\mathbf{b})$ and plane $\mathbf{r}\cdot\mathbf{n} = d$, then

$$\sin\theta = \frac{\mathbf{b}\cdot\mathbf{n}}{\mathbf{b}\,.|\mathbf{n}|}$$

(a) If the line is perpendicular to the plane, then it is parallel to the normal to the plane

$\therefore \mathbf{b} = \lambda\mathbf{n}$ for some scalar λ.

(b) If the line is parallel to the plane, then it is perpendicular to the normal to the plane

$\therefore \quad \mathbf{b}\cdot\mathbf{n} = 0$

(ii) **Cartesian form** If θ is the angle between the line $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ and the plane $ax + by + cz + d = 0$, then

$$\sin\theta = \frac{al + bm + cn}{\sqrt{a^2+b^2+c^2}\sqrt{l^2+m^2+n^2}}$$

(a) If the line is perpendicular to the plane, then it is parallel to its normal.

$$\therefore \quad \frac{l}{a} = \frac{m}{b} = \frac{n}{c}$$

(b) If the line is parallel to the plane, then it is perpendicular to its normal.

$\therefore \quad al + bm + cn = 0$

(17) CONDITIONS FOR A LINE TO LIE IN A PLANE

(i) **Vector form** If the line $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ lies in the plane $\mathbf{r}\cdot\mathbf{n} = d_1$, then

(a) $\mathbf{b}\cdot\mathbf{n} = 0$ and (b) $\mathbf{a}\cdot\mathbf{n} = d$

(ii) **Cartesian form** If the line $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ lies in the plane $ax + by + cz + d = 0$, then

(a) $ax_1 + by_1 + cz_1 + d = 0$ and

(b) $al + bm + cn = 0$

(18) COPLANARITY OF TWO LINES

(i) **Vector form** If the lines $\mathbf{r} = \mathbf{a}_1 + \lambda\mathbf{b}_1$ and $\mathbf{r} = \mathbf{a}_2 + \mu\mathbf{b}_2$ are coplanar, then $[\mathbf{a}_1 \mathbf{b}_1 \mathbf{b}_2] = [\mathbf{a}_2 \mathbf{b}_1 \mathbf{b}_2]$ and the equation of the plane containing them is

$$[\mathbf{r}\,\mathbf{b}_1\mathbf{b}_2] = [\mathbf{a}_1\mathbf{b}_1\mathbf{b}_2]$$

or $$[\mathbf{r}\,\mathbf{b}_1\mathbf{b}_2] = [\mathbf{a}_2\mathbf{b}_1\mathbf{b}_2]$$

(ii) **Cartesian form** If the lines $\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1}$ and $\frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2}$ are coplanar,

then $$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

and the equation of the plane containing them is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

or $$\begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

@ CLASS XII SYLLABUS

Rapid CONCEPT REVISION

TARGET 2017

MATRIX

A set of mn numbers (real or imaginary) arranged in the form of a rectangular array of m rows and n columns is called an $m \times n$ matrix (to be read as m by n matrix).

Note For a matrix having m rows and n columns, $m \times n$ is known as order and number of elements is equal to actual multiplication of m and n.

Trace of a Matrix

Let $A = [a_{ij}]_{n \times n}$ be a square matrix. Then, the sum of all diagonals elements of A is called the trace of A and is denoted by $\text{tr}(A)$.

Thus, $$\text{tr}(A) = \sum_{i=1}^{n} a_{ii}$$

Positive Integral Powers of a Square Matrix

Let A be a square matrix. Then, we define

(i) $A^1 = A$

(ii) $A^{n+1} = A^n A$, where $n \in N$

(iii) $A^m \cdot A^n = A^{m+n}$

(iv) $(A^m)^n = A^{mn}$

(v) If $A = [d_1\ d_2\ d_3 \ldots d_n]$ is a diagonal matrix, then $A^k = [d_1^k\ d_2^k\ d_3^k \ldots d_n^k]$, where $k \in N$

Idempotent Matrix

A square matrix A is called an idempotent matrix, if $A^2 = A$.

Nilpotent Matrix

A square matrix A is said to be a nilpotent matrix of order n, if n is the smallest positive integer such that $A^n = 0$.

Transpose of a Matrix

Let $A = [a_{ij}]$ be an $m \times n$ matrix. Then, the transpose of A, denoted by A^T or A', is an $n \times m$ matrix such that

$$(A^T)_{ij} = a_{ji},\ \forall\ i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n$$

Thus, A^T is obtained from A by changing its rows into columns or its columns into rows.

Properties of Transpose

Let A and B be two matrices. Then,

(i) $(A^T)^T = A$

(ii) $(A + B)^T = A^T + B^T$

(iii) $(kA)^T = kA^T$, k be any scalar.

(iv) $(AB)^T = B^T A^T$

Symmetric Matrix

A square matrix $A = [a_{ij}]$ is called a symmetric matrix, if

$$a_{ij} = a_{ji},\ \forall\ i, j \ \text{ or } \ A = A^T$$

Skew-symmetric Matrix

A square matrix $A = [a_{ij}]$ is a skew-symmetric matrix, if

$$a_{ij} = -a_{ji},\ \forall\ i, j \ \text{ or } \ A^T = -A$$

Some Useful Properties of Symmetric and Skew-symmetric Matrices

Following are some properties of symmetric and skew-symmetric matrices.

Property 1 All diagonal elements of a skew-symmetric matrix are zero.

Property 2 Every square matrix A can be uniquely expressed as the sum of a symmetric matrix P and a skew-symmetric matrix Q, where $P = \frac{1}{2}(A + A^T)$ and $Q = \frac{1}{2}(A - A^T)$.

Property 3 For any square matrix A, matrices AA^T and A^TA are symmetric matrices.

Property 4 If A and B are symmetric matrices, then AB is a symmetric matrix iff $AB = BA$.

Property 5 All positive integral powers of a symmetric matrix are symmetric, because $(A^n)^T = (A^T)^n$.

Property 6 All positive odd integral powers of a skew-symmetric matrix are skew-symmetric and positive even integral powers of a skew-symmetric matrix are symmetric, because $(A^n)^T = (A^T)^n$.

Determinant of a Square Matrix

Every square matrix can be associated to an expression or a number which is known as its determinant. If $A = [a_{ij}]$ is a square matrix of order n, then the determinant of A is denoted by det (A) or $|A|$.

Some Useful Results

(i) The determinant of the identity matrix is 1.
(ii) The determinant of a diagonal matrix is the product of its diagonal elements.
(iii) The determinant of skew-symmetric matrix of odd order is zero.
(iv) The determinant of a skew-symmetric matrix of even order is a perfect square.
(v) The determinant of a triangular matrix is equal to the product of its diagonal elements.
(vi) If A is a square matrix and $n \in N$, then $|A^n| = |A|^n$.

Singular Matrix

A square matrix is a singular matrix, if its determinant is zero. Otherwise, it is a non-singular matrix.

Adjoint of a Square Matrix

Let $A = [a_{ij}]$ be a square matrix of order n and let C_{ij} be cofactor of a_{ij} in A. Then, the transpose of the matrix of cofactor of elements of A is called the adjoint of A and is denoted by adj (A).

Thus, $\text{adj}(A) = [C_{ij}]^T$

$\Rightarrow$ $(\text{adj } A)_{ij} = C_{ji} = \text{cofactors of } a_{ji} \text{ in } A.$

Note The adjoint of a square matrix of order 2 is obtained by interchanging the diagonal elements and changing signs of off diagonal elements.

Inverse of a Matrix

A square matrix of order n is invertible, if there exists a square matrix B of the same order such that $AB = I_n = BA$

In such a case, we say that the inverse of A is B and $A^{-1} = B$.

Some Useful Results on Adjoint and Inverse of a Matrix

1. A square matrix is invertible, if it is non-singular.
   Also, $A^{-1} = \frac{1}{|A|}\text{adj}(A)$
2. Let A, B, C be square matrices of the same order n. If A is a non-singular matrix, then
   (i) $AB = AC \Rightarrow B = C$ [left cancellation law]
   (ii) $BA = CA \Rightarrow B = C$ [right cancellation law]
3. If A and B are invertible matrices of the same order, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.
4. If A is an invertible square matrix, then A^T is also invertible and $(A^T)^{-1} = (A^{-1})^T$.
5. Let A be a non-singular square matrix of order n. Then, $|\text{adj } A| = |A|^{n-1}$
6. If A and B are non-singular square matrices of the same order, then $\text{adj}(AB) = (\text{adj } B)(\text{adj } A)$
7. If A is an invertible square matrix, then $\text{adj}(A^T) = (\text{adj } A)^T$
8. If A is a non-singular square matrix, then $\text{adj}(\text{adj } A) = |A|^{n-2}A$
9. If a non-singular matrix A is symmetric, then A^{-1} is also symmetric.
10. If A is a non-singular matrix, then $|A^{-1}| = |A|^{-1}$, i.e. $|A^{-1}| = \frac{1}{|A|}$

Orthogonal Matrix

A square matrix A is called an orthogonal matrix, if

$$AA^T = A^TA = I$$

Note
- If A is an orthogonal matrix, then $|A| = \pm 1$.
- If A, B are orthogonal matrices, then AB and BA are also orthogonal matrices.
- If A is an orthogonal matrix, then A^T and A^{-1} are also orthogonal.

Rank of a Matrix

A number r is said to be the rank of an $m \times n$ matrix A, if every square submatrix of order $(r + 1)$ or more is singular. There exists atleast one square submatrix of order r, which is non-singular.

Some Theorems on Rank of a Matrix

Theorem 1 The rank of a matrix is same as the rank of its transpose, i.e. $r(A) = r(A^T)$.

Theorem 2 Elementary transformation do not alter the rank of matrix.

Consistency of a System of Linear Equation

Let $AX = B$ be a system of n-linear equations in n unknowns.

(i) If $|A| \neq 0$, then the system is consistent and has the unique solution given by $X = A^{-1}B$.

(ii) If $|A| = 0$ and (adj A) $B = 0$, then the system is consistent and has infinitely many solutions.

(iii) If $|A| = 0$ and (adj A)$B \neq 0$, then the system is inconsistent.

MASTER STROKES

1. If $A = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$, then the value of $|\text{adj } A|$, is

(a) a^{27} (b) a^9 (c) a^6 (d) a^2

2. If B is a non-singular matrix and A is a square matrix, then det $(B^{-1}AB)$ is equal to

(a) det (A) (b) det (B) (c) det (B^{-1}) (d) det (A^{-1})

3. The system of equations

$x + y + z = 5, \quad x + 2y + 3z = 9, \quad x + 3y + \lambda z = \mu$

has a unique solution, if

(a) $\lambda = 5, \mu = 13$ (b) $\lambda \neq 5$

(c) $\lambda = 5, \mu \neq 13$ (d) $\mu \neq 13$

4. If A and B are square matrices such that $B = -A^{-1}BA$, then $(A + B)^2$ is equal to

(a) 0 (b) $A^2 + B^2$

(c) $A^2 + 2AB + B^2$ (d) $A + B$

5. If $P = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$, $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $Q = PAP^T$, then $P^TQ^{2005}P$, is

(a) $\begin{bmatrix} 1 & 2005 \\ 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2005 \\ 2005 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 \\ 2005 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

6. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$ and U_1, U_2, U_3 be column matrices satisfying $AU_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $AU_2 = \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix}$, $AU_3 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$. If U is 3×3 matrix, whose columns are U_1, U_2, U_3, then $|U|$ is equal to

(a) 3 (b) –3 (c) $\frac{3}{2}$ (d) 2

7. Let A be the set of all 3×3 symmetric matrices whose enteries are either 0 or 1. The number of elements in set A, is

(a) 2^3 (b) 2^6 (c) 2^9 (d) 18

8. Let A and B be 3×3 matrices of real numbers, where A is symmetric, B is skew-symmetric and $(A + B)(A - B) = (A - B)(A + B)$. If $(AB)^T = (-1)^n AB$, then

(a) $n \in Z$

(b) $n \in N$

(c) n is an even natural number

(d) n is an odd natural number

9. Let k be a positive real number and let

$A = \begin{bmatrix} 2k-1 & 2\sqrt{k} & 2\sqrt{k} \\ 2\sqrt{k} & 1 & -2k \\ -2\sqrt{k} & 2k & -1 \end{bmatrix}$ and

$B = \begin{bmatrix} 0 & 2k-1 & \sqrt{k} \\ 1-2k & 0 & 2\sqrt{k} \\ -\sqrt{k} & -2\sqrt{k} & 0 \end{bmatrix}$

If det (adj A) + det (adj B) $= 10^6$, then $[k]$ is equal to ($[\cdot]$ denotes the greatest integer function)

(a) 3 (b) 4 (c) 5 (d) 6

10. Let P be an odd prime number and T_P be the following set of 2×2 matrices

$T_P = \left\{ A = \begin{bmatrix} a & b \\ c & a \end{bmatrix} : a, b, c \in \{0, 1, 2, \ldots, (p-1)\} \right\}$

The number of A in T_P such that A is either symmetric or skew-symmetric or both and det(A) divisible by p is

(a) $(p-1)^2$ (b) $2(p-1)$

(c) $(p-1)^2 + 1$ (d) $2p - 1$

11. Let ω $(\neq 1)$ be a cube root of unity and S be the set of all non-singular matrices of the form $\begin{bmatrix} 1 & a & b \\ \omega & 1 & c \\ \omega^2 & \omega & 1 \end{bmatrix}$ where, each of a, b and c is either ω or ω^2. Then, the number of distinct matrices in the set S is

(a) 2 (b) 6 (c) 4 (d) 8

12. If P is a 3×3 matrix such that $P^T = 2P + I$, where P^T is the transpose of P and I is 3×3 identity, then there exists a column matrix $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ such that PX is equal to

(a) $\begin{bmatrix}0\\0\\0\end{bmatrix}$ (b) X (c) $2X$ (d) $-X$

13. Let $P=\begin{bmatrix}\cos\frac{\pi}{9} & \sin\frac{\pi}{9}\\ -\sin\frac{\pi}{9} & \cos\frac{\pi}{9}\end{bmatrix}$ and α, β, γ be non-zero real numbers such that $\alpha P^6 + \beta P^3 + \gamma I$ is the zero matrix.

Then, $(\alpha^2+\beta^2+\gamma^2)^{(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)}$ is

(a) π (b) $\frac{\pi}{2}$ (c) 0 (d) 1

14. The number of values of k for which the system of equations

$$(k+1)x+8y=4k; \quad kx+(k+3)y=3k-1$$

has no solution, is

(a) infinite (b) 1 (c) 2 (d) 3

15. Let ω be a complex cube root of unity with $\omega \neq 1$ and $P=[p_{ij}]$ be a $n\times n$ matrix with $p_{ij}=\omega^{i+j}$. If $p^2=0$, then n can take value

(a) 57 (b) 55 (c) 58 (d) 56

16. If A is an 3×3 non-singular matrix such that $AA^T=A^TA$ and $B=A^{-1}A^T$, then BB^T equals

(a) B^{-1} (b) $(B^{-1})^T$ (c) $I+B$ (d) I

17. If $S_k=\begin{bmatrix}1 & k\\ 0 & 1\end{bmatrix}$, $k\in N^+$, where N is the set of all natural numbers, then $(S_2)^n(S_k)^{-1}$ for $n\in N$ is

(a) S_{2n-k} (b) S_{2n+k-1}
(c) S_{2^n+k-1} (d) S_{2^n+k+1}

18. Let M and N be two 3×3 matrices such that $MN=NM$. Further, if $M\neq N^2$ and $M^2=N^4$, then

(a) determination of (M^2+MN^2) is zero
(b) there is a 3×3 non-zero matrix U such that $(M^2+MN^2)U$ is the zero matrix
(c) determinant of $(M^2+MN^2)\geq 1$
(d) for a 3×3 matrix U, if $(M^2+MN^2)U$ equals the zero matrix, then U is the zero matrix

19. Let M be a 2×2 symmetric matrix with integer entries. Then, M is invertible, if

(a) the first column of M is the transpose of the second row of M
(b) the second column of M is the transpose of the first column of M
(c) M is a diagonal matrix with non-zero entries in the main diagonal
(d) the product of entries in the main diagonal of M is not the square of an integer

20. Let X and Y be two arbitrary 3×3 non-zero skew-symmetric matrices and Z be an arbitrary 3×3 non-zero symmetric matrix. Then, which of the following matrices is(are) skew-symmetric?

(a) $Y^3Z^4-Z^4Y^3$ (b) $X^{44}+Y^{44}$
(c) $X^4Z^3-Z^3X^4$ (d) $X^{23}+Y^{23}$

21. If $A=[a_{ij}]_{n\times n}$, where $a_{ij}=i^{100}+j^{100}$, then $\lim\limits_{n\to\infty}\left(\frac{\sum\limits_{i=1}^{n}a_{ij}}{n^{101}}\right)$ equals to

(a) $\frac{1}{50}$ (b) $\frac{1}{101}$
(c) $\frac{2}{101}$ (d) $\frac{3}{101}$

22. $A_{n\times n}$ matrix is formed using 0, 1 and –1 as its elements. The number of such matrices which are skew-symmetric is

(a) $\frac{n(n-1)}{2}$ (b) $(n-1)^2$
(c) $2^{\frac{n(n-1)}{2}}$ (d) $3^{\frac{n(n-1)}{2}}$

23. The number of elements that a square matrix of order n has below its leading diagonal is

(a) $\frac{n(n+1)}{2}$ (b) $\frac{n(n-1)}{2}$
(c) $\frac{(n-1)(n-2)}{2}$ (d) $\frac{(n+1)(n+2)}{2}$

24. If $A=[a_{ij}]_{4\times 4}$ such that $a_{ij}=\begin{cases}2, & i=j\\ 0, & i\neq j\end{cases}$, then $\left\{\frac{1}{7}\det(\text{adj}(\text{adj}\,A))\right\}$ is, where $\{\cdot\}$ denotes the fractional part function.

(a) $\frac{1}{7}$ (b) $\frac{2}{7}$
(c) $\frac{3}{7}$ (d) None of these

25. If A is skew-symmetric matrix of order 2 and B, C are matrices $\begin{bmatrix}1 & 4\\ 2 & 9\end{bmatrix}$ and $\begin{bmatrix}9 & -4\\ -2 & 1\end{bmatrix}$ respectively.

Then,

$A^3BC+A^5(B^2C^2)+A^7(B^3C^3)+\ldots+A^{2n+1}B^nC^n$ is

(a) a symmetric matrix
(b) a skew-symmetric matrix
(c) an identity matrix
(d) None of the above

Answers

1. (c)	**2.** (a)	**3.** (b)	**4.** (b)	**5.** (a)
6. (a)	**7.** (b)	**8.** (d)	**9.** (b)	**10.** (c)
11. (a)	**12.** (d)	**13.** (d)	**14.** (b)	**15.** (a)
16. (d)	**17.** (a)	**18.** (a,b)	**19.** (c,d)	**20.** (c,d)
21. (c)	**22.** (d)	**23.** (b)	**24.** (a)	**25.** (b)

DETERMINANT

Every square matrix can be associated to an expression or a number which is known as its determinant. For a matrix A its determinant value is denoted by $\det(A)$ or $|A|$.

Minor

Let $A=[a_{ij}]$ be a square matrix of order n. Then, the minor M_{ij} of a_{ij} in A is the determinant of the square submatrix of order $(n-1)$ obtained by leaving ith row and jth column of A.

Cofactors

Let $A=[a_{ij}]$ be a square matrix of order n. Then, the cofactor C_{ij} of a_{ij} in A is equal to $(-1)^{i+j}$ times the determinant of the submatrix of order $(n-1)$ obtained by leaving ith row and jth column of A.

Thus, $C_{ij}=(-1)^{i+j}\,M_{ij}$

or $C_{ij}=\begin{cases} M_{ij} & , \text{ if } i+j \text{ is even} \\ -M_{ij} & , \text{ if } i+j \text{ is odd} \end{cases}$

Properties of Determinant

Following are some properties of determinants

Property 1 Let $A=[a_{ij}]$ be a square matrix of order n, then the sum of the product of elements of any row (column) with their cofactors is always equal to $|A|$ or $\det(A)$.

i.e. $\sum_{j=1}^{n} a_{ij}\,C_{ij}=|A|$ and $\sum_{i=1}^{n} a_{ij}\,C_{ij}=|A|$

Property 2 Let $A=[a_{ij}]$ be a square matrix of order n, then sum of the product of elements of any row (column) with the cofactors of the corresponding elements of some other row (column) is zero.

i.e. $\sum_{j=1}^{n} a_{ij}\,C_{kj}=0$ and $\sum_{i=1}^{n} a_{ij}\,C_{ik}=0$

Property 3 Let $A=[a_{ij}]$ be a square matrix of order n, then $|A|=|A^T|$. By the abuse of language this property is also stated as follows

The value of a determinant remains unchanged, if its rows and columns are interchanged.

Property 4 Let $A=[a_{ij}]$ be a square matrix of order $n\ (\geq 2)$ and let B be a matrix obtained from A by interchanging any two rows (columns) of A, then $|B|=-|A|$.

Conventionally, this property is also stated as follows

If any two rows (columns) of a determinant are interchanged, then the value of the determinant changes by minus sign only.

Property 5 If any two rows (columns) of a square matrix $A=[a_{ij}]$ of order $n\ (\geq 2)$ are identical, then its determinant is zero, i.e. $|A|=0$.

Conventionally,this property is also stated as follows

If any two rows or columns of a determinant are identical, then its value is zero.

Property 6 Let $A=[a_{ij}]$ be a square matrix of order n and let B be the matrix obtained from A by multiplying each element of a row (column) of A by a scalar k, then $|B|=k|A|$.

Conventionally, this property is also stated as follows

If each element of a row (column) of a determinant is multiplied by a constant k, then the value of the new determinant is k times the value of the original determinant.

Deduction Let $A=[a_{ij}]$ be a square matrix of order n, then $|kA|=k^n|A|$

Property 7 Let A be a square matrix such that each element of a row (column) of A is expressed as the sum of two or more terms. Then, the determinant of A can be expressed as the sum of the determinants of two or more matrices of the same order.

Conventionally, this property is also stated as follows

If each element of a row (column) of a determinant is expressed as a sum of two or more terms, then the determinant can be expressed as the sum of two or more determinants.

Property 8 Let A be a square matrix and B be a matrix obtained from A by adding to a row (column) of A a scalar multiple of another row (column) of A, then $|B|=|A|$.

This property is conventionally stated as follows

If each element of a row (column) of a determinant is multiplied by the same constant and then added to the corresponding elements of some other row (column), then the value of the determinant remains same.

Property 9 Let A be a square matrix of order $n\ (\geq 2)$ such that each element in a row (column) of A is zero,then $|A|=0$.

Conventionally, this property is also stated as follows

If each element of a row (column) of a determinant is zero, then its value is zero.

Property 10 If $A=[a_{ij}]$ is a diagonal matrix of order $n\ (\geq 2)$, then $|A|=a_{11}\times a_{22}\times a_{33}\times\ldots\times a_{nn}$

Property 11 If A and B are square matrices of the same order, then $|AB|=|A||B|$

Property 12

(i) The determinant of a skew-symmetric matrix of odd order is zero.

(ii) The determinant of a skew-symmetric matrix of even order is a perfect square.

Product of Determinants

Let $\Delta_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $\Delta_2 = \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$ be two determinants.

Then, $\Delta_1 \Delta_2 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \cdot \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$

$$= \begin{vmatrix} a_1\ \alpha_1 + b_1\ \alpha_2 + c_1\ \alpha_3 & a_1\ \beta_1 + b_1\ \beta_2 + c_1\ \beta_3 & a_1\ \gamma_1 + b_1\ \gamma_2 + c_1\ \gamma_3 \\ a_2\ \alpha_1 + b_2\ \alpha_2 + c_2\ \alpha_3 & a_2\ \beta_1 + b_2\ \beta_2 + c_2\ \beta_3 & a_2\ \gamma_1 + b_2\ \gamma_2 + c_2\ \gamma_3 \\ a_3\ \alpha_1 + b_3\ \alpha_2 + c_3\ \alpha_3 & a_3\ \beta_1 + b_3\ \beta_2 + c_3\ \beta_3 & a_3\ \gamma_1 + b_3\ \gamma_2 + c_3\ \gamma_3 \end{vmatrix}$$

This is row-by-column multiplication rule for finding the product of two determinants and it is same as the rule of multiplication of two matrices. We know that the value of a determinant does not alter by interchanging the rows and columns. So, we can also follow the row-by-row or the column-by-row or column-by-column multiplication rule.

Differentiation of Determinants

Let $\Delta(x) = \begin{vmatrix} f_1(x) & g_1(x) \\ f_2(x) & g_2(x) \end{vmatrix}$, where $f_1(x)$, $f_2(x)$, $g_1(x)$ and $g_2(x)$ are functions of x. Then, $\Delta'(x) = \begin{vmatrix} f_1'(x) & g_1'(x) \\ f_2(x) & g_2(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & g_1(x) \\ f_2'(x) & g_2'(x) \end{vmatrix}$

Also, $\Delta'(x) = \begin{vmatrix} f_1'(x) & g_1(x) \\ f_2'(x) & g_2(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & g_1'(x) \\ f_2(x) & g_2'(x) \end{vmatrix}$

Thus, to differentiate a determinant, we differentiate one row (or column) at a time, keeping others unchanged.

If we write $\Delta(x) = [C_1, C_2]$, where C_i denotes the ith column, then $\Delta'(x) = [C_1' \; C_2] + [C_1 \; C_2']$, where C_i' denotes the column obtained by differentiating functions in the ith column C_i.

Also, if $\Delta(x) = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}$, then $\Delta'(x) = \begin{bmatrix} R'_1 \\ R_2 \end{bmatrix} + \begin{bmatrix} R_1 \\ R'_2 \end{bmatrix}$

Similarly, we can differentiate determinants of higher order.

Applications of Determinants

Area of a Triangle

The area of a triangle having vertices at $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) is the absolute value of Δ given by

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

If points $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) are collinear, then

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

The equation of the line joining points (x_1, y_1) and (y_2, y_2) is given by

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

System of Linear Equations

Consider, a system of simultaneous linear equations given by

$$\left.\begin{aligned} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{aligned}\right\} \quad \ldots(i)$$

A set of values of the variables x, y, z which simultaneously satisfy these three equations is called a solution. A system of linear equations may have a unique solution, or many solutions, or no solution at all. If it has a solution (whether unique or not) the system is said to be **consistent**. If it has no solution, it is called an **inconsistent** system.

If $d_1 = d_2 = d_3 = 0$ in Eq. (i). Then, the system of equations is said to be a homogeneous system. Otherwise, it is called a non-homogeneous system of equations.

Theorem 1 (Cramer's rule) The solution of the system of simultaneous linear equations

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

is given by $x = \frac{D_1}{D}$, $y = \frac{D_2}{D}$, where

$$D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, D_1 = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} \text{ and } D_2 = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

provided that $D \neq 0$.

(i) If $D \neq 0$, then the given system of equations is consistent and has a unique solution given by $x = \frac{D_1}{D}, y = \frac{D_2}{D}$

(ii) If $D = 0$ and $D_1 = D_2 = 0$, then the system is consistent and has infinitely many solutions.

(iii) If $D = 0$ and one of D_1 and D_2 is non-zero, then the system is inconsistent.

Theorem 2 If

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

is a system of simultaneous linear equations, then its solution is given by

$$x = \frac{D_1}{D}, y = \frac{D_2}{D} \text{ and } z = \frac{D_3}{D}, \text{ where}$$

$$D=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},$$

$$D_1=\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix},$$

$$D_2=\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$$

and $D_3=\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$, provided that $D \neq 0$.

(i) If $D \neq 0$, then the given system of equations is consistent and has a unique solution given by

$$x=\frac{D_1}{D},\ y=\frac{D_2}{D} \quad \text{and} \quad z=\frac{D_3}{D}$$

(ii) If $D = 0$ and $D_1 = D_2 = D_3 = 0$, then the given system of equations is consistent with infinitely many solutions.

(iii) If $D = 0$ and atleast one of the determinant D_1, D_2, D_3 is non-zero, then the given system of equations is inconsistent.

If
$$a_1x + b_1y + c_1z = 0$$
$$a_2x + b_2y + c_2z = 0$$
$$a_3x + b_3y + c_3z = 0$$
is a homogeneous system of equations, such that

$$D=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0,$$

then $x = y = z = 0$ is the only solution and it is known as the trivial solution.

If $D = 0$, then the system is consistent with infinitely many solutions.

MASTER STROKES

1. If $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 5$, then the value of

$$\Delta=\begin{vmatrix} b_2c_3-b_3c_2 & a_3c_2-a_2c_3 & a_2b_3-a_3b_2 \\ b_3c_1-b_1c_3 & a_1c_3-a_3c_1 & a_3b_1-a_1b_3 \\ b_1c_2-b_2c_1 & a_2c_1-a_1c_2 & a_1b_2-a_2b_1 \end{vmatrix} \text{ is}$$

(a) 5 *(b)* 25 *(c)* 125 *(d)* 0

2. If $\Delta_r=\begin{vmatrix} 2^{r-1} & 2\cdot 3^{r-1} & 4\cdot 5^{r-1} \\ x & y & z \\ 2^{n-1} & 3^{n-1} & 5^{n-1} \end{vmatrix}$, then $\sum_{r=1}^{n}\Delta_r$ is equal to

(a) xyz *(b)* 1 *(c)* -1 *(d)* 0

3. The value of

$$\begin{vmatrix} 2y_1z_1 & y_1z_2+y_2z_1 & y_1z_3+y_3z_1 \\ y_1z_2+y_2z_1 & 2y_2z_2 & y_2z_3+y_3z_2 \\ y_1z_3+y_3z_1 & y_2z_3+y_3z_2 & 2y_3z_3 \end{vmatrix} \text{ is}$$

(a) 0 *(b)* $y_1y_2y_3z_1z_2z_3$
(c) $y_1+y_2+y_3$ *(d)* $z_1+z_2+z_3$

4. Let f, g and h be differentiable functions of x, if

$$\Delta=\begin{vmatrix} f & g & h \\ (xf)' & (xg)' & (xh)' \\ (x^2f)'' & (x^2g)'' & (x^2h)'' \end{vmatrix} \text{ and}$$

$$\Delta'=\begin{vmatrix} f & g & h \\ f' & g' & h' \\ (x^nf'')' & (x^ng'')' & (x^nh'')' \end{vmatrix}, \text{ then } n \text{ is equal to}$$

(a) 1 *(b)* 2 *(c)* 3 *(d)* 4

5. If $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are vertices of an equilateral triangle whose each side is equal to a, then $\begin{vmatrix} x_1 & y_1 & 2 \\ x_2 & y_2 & 2 \\ x_3 & y_3 & 2 \end{vmatrix}^2$ is equal to

(a) $2a^2$ *(b)* $2a^4$ *(c)* $3a^2$ *(d)* $3a^4$

6. If $a > b > c$ and the system of equations $ax + by + cz = 0$, $bx + cy + az = 0$ and $cx + ay + bz = 0$ has a non-trivial solution, then the quadratic equation $ax^2 + bx + c = 0$ has

(a) atleast one positive root
(b) roots opposite in sign
(c) positive roots
(d) imaginary roots

7. If α, β and γ are the roots of $x^3 + ax^2 + b = 0$, then the value of

$$\begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix} \text{ is}$$

(a) $-a^3$ *(b)* a^3-3b *(c)* a^3 *(d)* a^2-3b

8. If

$$\begin{vmatrix} x^n & x^{n+2} & x^{n+3} \\ y^n & y^{n+2} & y^{n+3} \\ z^n & z^{n+2} & z^{n+3} \end{vmatrix} = (x-y)(y-z)(z-x)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right),$$

then n is equal to

(a) 1 *(b)* -1 *(c)* 2 *(d)* -2

9. Let $\begin{vmatrix} x & 2 & x \\ x^2 & x & 6 \\ x & x & 6 \end{vmatrix} = ax^4 + bx^3 + cx^2 + dx + e$. Then, the value of $5a + 4b + 3c + 2d + e$ is equal to

(a) 0 *(b)* -16

(c) 16 *(d)* None of these

10. If the system of linear equations

$$x + 4ay + az = 0$$
$$x + 3by + bz = 0$$
$$x + 2cy + cz = 0$$

have a non-trivial solution, then a, b and c are in

(a) HP *(b)* GP

(c) AP *(d)* None of these

11. If x, y and z are positive numbers such that y and z have respectively 1 and 0 at their unit's place and Δ is the determinant $\begin{vmatrix} x & 4 & 1 \\ y & 0 & 1 \\ z & 1 & 0 \end{vmatrix}$.

If $(\Delta + 1)$ is divisible by 10, then x has at its unit's place

(a) 1 *(b)* 0

(c) 2 *(d)* None of these

12. If $x^a\ y^b = e^m, x^c\ y^d = e^n, \Delta_1 = \begin{vmatrix} m & b \\ n & d \end{vmatrix}, \Delta_2 = \begin{vmatrix} a & m \\ c & n \end{vmatrix}$ and $\Delta_3 = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$, then the values of x and y are

(a) $\frac{\Delta_1}{\Delta_3}$ and $\frac{\Delta_2}{\Delta_3}$ *(b)* $\frac{\Delta_2}{\Delta_1}$ and $\frac{\Delta_3}{\Delta_1}$

(c) $\log\left(\frac{\Delta_1}{\Delta_3}\right)$ and $\log\left(\frac{\Delta_2}{\Delta_3}\right)$ *(d)* e^{Δ_1/Δ_3} and e^{Δ_2/Δ_3}

13. The number of 3×3 non-singular matrices with four enteries as 1 and all other enteries as 0, is

(a) 6 *(b)* atleast 7

(c) less than 4 *(d)* 5

14. If $f(\theta) = \begin{vmatrix} 1 & \tan\theta & 1 \\ -\tan\theta & 1 & \tan\theta \\ -1 & -\tan\theta & 1 \end{vmatrix}$, then the set $\{ f(\theta) : 0 \le \theta \le \frac{\pi}{2} \}$ is

(a) $(-\infty, 0] \cup [2, \infty)$ *(b)* $[2, \infty)$

(c) $(-\infty, 0) \cup (0, \infty)$ *(d)* $(-\infty, -1] \cup [1, \infty)$

15. Let $P = [a_{ij}]$ be a 3×3 matrix and let $Q = [b_{ij}]$, where $b_{ij} = 2^{i+j} a_{ij}$ for $1 \le i, j \le 3$. If the determinant of P is 2, then the determinant of the matrix Q is

(a) 2^{10} *(b)* 2^{11}

(c) 2^{12} *(d)* 2^{13}

16. In a ΔABC, if $\begin{vmatrix} 1 & a & b \\ 1 & c & a \\ 1 & b & c \end{vmatrix} = 0$, then $\sin^2 A + \sin^2 B + \sin^2 C$ is equal to

(a) $\frac{3\sqrt{3}}{2}$ *(b)* $\frac{9}{4}$ *(c)* $\frac{5}{4}$ *(d)* 2

17. Which of the following values of α satisfy the equation

$$\begin{vmatrix} (1+\alpha)^2 & (1+2\alpha)^2 & (1+3\alpha)^2 \\ (2+\alpha)^2 & (2+2\alpha)^2 & (2+3\alpha)^2 \\ (3+\alpha)^2 & (3+2\alpha)^2 & (3+3\alpha)^2 \end{vmatrix} = -648\alpha \ ?$$

(a) -4 *(b)* 9

(c) 8 *(d)* -8

18. The set of the all values of λ for which system of linear equations

$2x_1 - 2x_2 + x_3 = \lambda x_1$, $2x_1 - 3x_2 + 2x_3 = \lambda x_2$ and $-x_1 + 2x_2 = \lambda x_3$ has a non-trivial solution

(a) contains two elements

(b) contains more than two elements

(c) is an empty set

(d) is a singleton set

19. Let $A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ and let A_1 denotes the matrix of the cofactors of elements of matrix A and A_2 be the matrix of cofactors of elements of matrix A_1 and so on. If A_n denotes the matrix of cofactors of elements of matrix A_{n-1}, then $|A_n|$ equals

(a) $|A|^{2n}$ *(b)* $|A|^{2^n}$

(c) $|A|^{n^2}$ *(d)* $|A|^2$

20. If $a^2 + b^2 + c^2 + ab + bc + ca \le 0$, $\forall\ a, b, c \in R$, then the value of the determinant

$$\begin{vmatrix} (a+b+2)^2 & a^2+b^2 & 1 \\ 1 & (b+c+2)^2 & b^2+c^2 \\ c^2+a^2 & 1 & (c+a+2)^2 \end{vmatrix}$$

is equal to

(a) 65 *(b)* $a^2 + b^2 + c^2 + 31$

(c) $4(a^2 + b^2 + c^2)$ *(d)* 0

21. Let $D_r = \begin{vmatrix} a & 2^r & 2^{16} - 1 \\ b & 3(4)^r & 2(4^{16} - 1) \\ c & 7(8^r) & 4(8^{16} - 1) \end{vmatrix}$, then the value of $\sum_{k=1}^{16} D_k$, is

(a) 0

(b) $a + b + c$

(c) $ab + bc + ca$

(d) None of the above

22. Let $\{D_1, D_2, D_3, \ldots, D_n\}$ be the set of all third order determinants that can be made with the distinct non-zero real numbers $a_1, a_2, \ldots, a_9$, then

(a) $\sum_{i=1}^{n} D_i = 1$ (b) $\sum_{i=1}^{n} D_i = 0$

(c) $D_i = D_j, \forall\, i, j$ (d) None of these

23. If $A = \int_1^{\sin\theta} \frac{t}{1+t^2}dt$ and $B = \int_1^{\operatorname{cosec}\theta} \frac{1}{t(1+t^2)}dt$, then the value of the determinant

$$\begin{vmatrix} A & A^2 & B \\ e^{A+B} & B^2 & -1 \\ 1 & A^2+B^2 & -1 \end{vmatrix}$$ is

(a) $\sin\theta$ (b) $\operatorname{cosec}\theta$ (c) 0 (d) 1

24. If the determinant

$$\begin{vmatrix} a & b & 2a\alpha + 3b \\ b & c & 2b\alpha + 3c \\ 2a\alpha + 3b & 2b\alpha + 3c & 0 \end{vmatrix} = 0,$$ then

(a) a, b, c are in HP

(b) α is a root of $4ax^2 + 12bx + 9c = 0$ or a, b, c are in GP

(c) a, b, c are in GP only

(d) a, b, c are in AP

25. If $a = 1 + 2 + 4 + \ldots$ n terms

$b = 1 + 3 + 9 + \ldots$ n terms

$c = 1 + 5 + 25 + \ldots$ n terms, then $\begin{vmatrix} a & 2b & 4c \\ 2 & 2 & 2 \\ 2^n & 3^n & 5^n \end{vmatrix}$ is equal to

(a) 30^n (b) 10^n (c) 0 (d) $2^n + 3^n + 5^n$

Answers

1. (b)	**2.** (d)	**3.** (a)	**4.** (c)	**5.** (d)
6. (a)	**7.** (c)	**8.** (b)	**9.** (d)	**10.** (a)
11. (c)	**12.** (d)	**13.** (b)	**14.** (b)	**15.** (d)
16. (b)	**17.** (b)	**18.** (a)	**19.** (b)	**20.** (a)
21. (a)	**22.** (b)	**23.** (c)	**24.** (b)	**25.** (c)

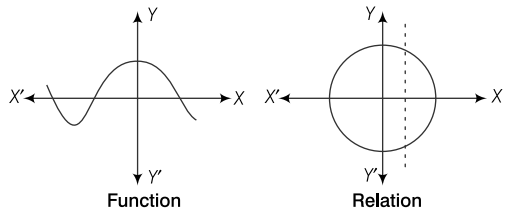
FUNCTION

A relation 'f' from a set A to a set B is called a function, if

(i) each element of A is associated with unique element of B, i.e. for each $a \in A$ there exists a unique $b \in B$ such that $(a, b) \in f$

(ii) each element of A has unique image in B, i.e. $(a, b) \in f$ and $(a, c) \in f \Rightarrow b = c$

Thus, the ordered pairs of f must satisfy the property that each element of A appears in some ordered pair and no two ordered pairs have same first element.



Note Every function is a relation but every relation is not necessarily function.

Domain and Range of Function

Let $f : A \to B$ be a function. Then, A is called domain, B is called co-domain and range of f is denoted by $f(A) = \{f(a) : a \in A\}$. $f(A)$ is also called the image set of A under the function f.

Note
- Range is a subset of co-domain. If only the rule of function is given, then the domain of the function is the set of those real numbers, where function is defined.
- For continuous function, the interval from minimum value to maximum value of a function gives the range.
- If domain of $f(x)$ is D_1 and domain of $g(x)$ is D_2, then domain of $f(x) + g(x) = D_1 \cap D_2$.

Real Valued Function

If the range of a function is subset of the real number R, then the function is called real valued function.

Algebra of Real Valued Function

Let f and g be real valued functions defined on a set A. Then, we define the real valued function $f + g$, $f - g$ and $f \cdot g$ on the set A.

(i) $(f + g)(x) = f(x) + g(x)$

(ii) $(f - g)(x) = f(x) - g(x)$

(iii) $(f \cdot g)(x) = f(x) \cdot g(x)$

(iv) If $g(x) \neq 0, \forall\, x \in A$, then $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$

(v) If n is a positive integer, then $f^n(x) = (f(x))^n$

Types of Functions

Polynomial Function

If a function f is defined by

$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n$,

where n is a non-negative integer and $a_0, a_1, a_2, \ldots, a_n$ are real numbers and $a_0 \neq 0$, then f is called a polynomial function of degree n.

Note
- A polynomial of degree one with no constant term is called an odd linear function, i.e. $f(x) = ax$, $a \neq 0$
- There are two polynomial functions, satisfying the relation $f(x) \cdot f(1/x) = f(x) + f(1/x)$. They are $f(x) = x^n + 1$ and $f(x) = 1 - x^n$, where n is a positive integer.

Algebraic Function

y is an algebraic function of x, if it is a function that satisfies an algebraic equation of the form $P_0(x)\,y^n + P_1(x)\,y^{n-1} + \ldots + P_{n-1}(x)\,y + P_n(x) = 0$, where n is a positive integer and $P_0(x), P_1(x), \ldots$ are polynomials in x, e.g. $x^3 + y^3 - 3xy = 0$ or $y = |x|$ is an algebraic function, since it satisfies the equation $y^2 - x^2 = 0$.

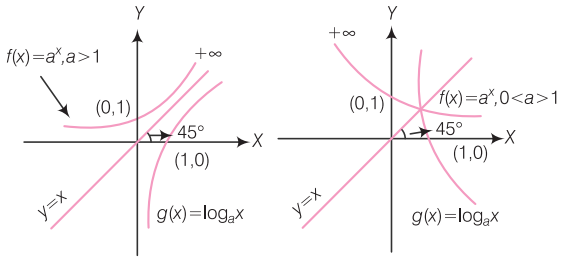
Note All polynomial functions are algebraic but not the converse. A function that is not algebraic is called transcedental function.

Rational Function

A rational function is a function of the form $y = f(x) = \dfrac{g(x)}{h(x)}$, where $g(x)$ and $h(x)$ are polynomial and $h(x) \neq 0$. Domain of $f(x)$ is $R - \{h(x) = 0\}$.

Exponential Function

A function $f(x) = a^x = e^{x \ln a} (a > 0,\ a \neq 1, x \in R)$ is called an exponential function. The inverse of the exponential function is called the logarithmic function.



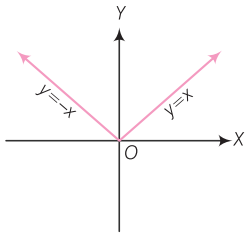
i.e. $g(x) = \log_a x$. For $f(x) = e^x$ domain is R and range is R^+.

For $f(x) = e^{\frac{1}{x}}$ domain is $R - \{0\}$ and range is $R^+ - \{1\}$.

Absolute Value Function

A function $y = f(x) = |x|$ is called the absolute value function or modulus function. It is defined as

$$y = |x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$



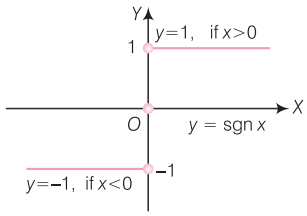
For $f(x) = |x|$, domain is R and range is $R^+ \cup \{0\}$.

For $f(x) = \dfrac{1}{|x|}$, domain is $R - \{0\}$ and range is R^+.

Signum Function

A function $y = f(x) = \text{sgn}\,(x)$ is defined as follows

$$y = f(x) = \begin{cases} 1, & \text{for } x > 0 \\ 0, & \text{for } x = 0 \\ -1, & \text{for } x < 0 \end{cases}$$



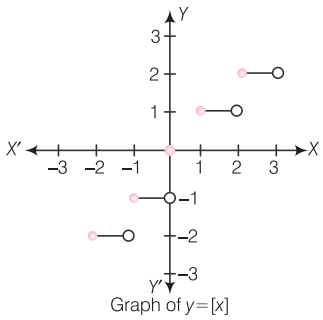
It is also written as $\text{sgn}\,(x) = |x|/x;\ x \neq 0;\ f(0) = 0$

Greatest Integer or Step up Function

The function $y = f(x) = [x]$ is called the greatest integer function, where $[x]$ denotes the greatest integer less than or equal to x.

Note that for $-1 \leq x < 0; [x] = -1,\ 0 \leq x < 1; [x] = 0$

$1 \leq x < 2; [x] = 1,\ 2 \leq x < 3; [x] = 2$ and so on.



Graph of $y = [x]$

For $f(x) = [x]$, domain is R and range is I.

For $f(x) = \dfrac{1}{[x]}$, domain is $R - [0, 1)$ and range is $\left\{\dfrac{1}{n} \middle| n \in I - \{0\}\right\}$.

Properties of Greatest Integer Function

(i) $[x] \leq x < [x] + 1$ and $x - 1 < [x] \leq x,\ 0 \leq x - [x] < 1$

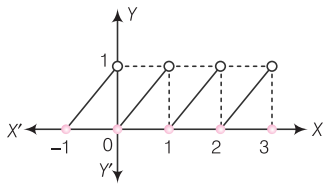
(ii) $[x + m] = [x] + m$, if m is an integer.

(iii) $[x] + [y] \leq [x + y] \leq [x] + [y] + 1$

(iv) $[x] + [-x] = \begin{cases} 0, \text{if } x \text{ is an integer} \\ -1, \text{otherwise} \end{cases}$

Fractional Part Function

It is defined by $g(x) = \{x\} = x - [x]$ e.g. The fractional part of the number 2.1 is $2.1 - 2 = 0.1$ and the fractional part of -3.7 is 0.3. The period of this function is 1 and graph of this function is as shown.



Graph of $y = \{x\}$

For $f(x) = \{x\}$, domain is R and range is $[0, 1)$.

For $f(x) = \dfrac{1}{\{x\}}$, domain is $R - I$ and range is $(1, \infty)$.

Identity Function

A function $f : A \to B$ is called identify function, if $f(x) = x$, $\forall\ x \in A$ and is denoted by I_A.

Constant Function

A function $f : A \to B$ is said to be a constant function. If every element of A has the same f image in B. Thus, $f : A \to B : f(x) = k$, $\forall\ x \in A$, $k \in B$ is a constant function.

Even and Odd Functions

(i) A function $f(x)$ defined on the symmetric interval $(-a, a)$. If $f(-x) = f(x)$ for all x in the domain of f, then f is said to be an even function.
e.g. $f(x) = \cos x$; $g(x) = x^2 + 3$

(ii) If $f(-x) = f(y)$ for all x in the domain of f, then f is said to be an odd function.
e.g. $f(x) = \sin x$; $g(x) = x^2 + x$

Note

- $f(x) - f(-x) = 0 \Rightarrow f(x)$ is even and $f(x) + f(-x) = 0 \Rightarrow f(x)$ is odd.
- A function may neither be odd nor even.
- Inverse of an even function is not defined and an even function cannot be strictly monotonic.
- Every even function is symmetric about the Y-axis and every odd function is symmetric about the origin.
- Every function can be expressed as the sum of an even and an odd function, e.g. $f(x) = \underbrace{\frac{f(x) + f(-x)}{2}}_{\text{Even}} + \underbrace{\frac{f(x) - f(-x)}{2}}_{\text{Odd}}$
- The only function which is defined on the entire number line and is even odd at the same time is $f(x) = 0$.
- If f and g both are even or odd, then the function $f \cdot g$ will be even but if anyone of them is odd, then $f \cdot g$ will be odd.

$f(x)$	$g(x)$	$f(x) + g(x)$	$f(x) - g(x)$
odd	odd	odd	odd
even	even	even	even
odd	even	neither odd nor even	neither odd nor even
even	odd	neither odd nor even	neither odd nor even
$f(x) \cdot g(x)$	$f(x) / g(x)$	$(gof)(x)$	$(fog)(x)$
even	even	odd	odd
even	even	even	even
odd	odd	even	even
odd	odd	even	even

Periodic Function

A function $f(x)$ is called periodic, if there exists a positive number T $(T > 0)$ called the period of the function such that $f(x + T) = f(x)$, for all values of x, within the domain of x.
e.g. $\sin(2\pi + x) = \sin x$ etc.

Note

- $f(T) = f(0) = f(-T)$, where T is the period.
- Inverse of a periodic function does not exist.
- Every constant function is always periodic, with no fundamental period.
- If $f(x)$ has a period T and $g(x)$ also has a period T, then it does not mean that $f(x) + g(x)$ must have a period T.
- If $f(x)$ has a period p, then $\frac{1}{f(x)}$ and $\sqrt{f(x)}$ also has a period p.
- If $f(x)$ has a period T, then $f(ax + b)$ has a period T/a $(a > 0)$.

Transformation of Graphs

The following table will help you to transform the graphs of functions

(i)	$y = f(x)$ to $y = -f(x)$	: Flip about the X-axis
(ii)	$y = f(x)$ to $y = f(-x)$	: Flip about the Y-axis
(iii)	$y = f(x)$ to $y = f(x) + k$	: Shift the graph $\lvert k \rvert$ units upwards or downwards depending on whether k is positive or negative, respectively.
(iv)	$y = f(x)$ to $y = f(x + k)$	: Advance (shift left) or delay (shift right) the graph by $\lvert k \rvert$ units depending on whether k is positive or negative, respectively.
(v)	$y = f(x)$ to $y = kf(x)$	: Stretch or compress the graph along the Y-axis depending on whether $\lvert k \rvert > 1$ or $\lvert k \rvert < 1$, respectively. Also, flip it about the X-axis, if k is negative (this latter statement follows from part i)
(vi)	$y = f(x)$ to $y = f(kx)$	: Stretch or compress the graph along the X-axis depending on whether $\lvert k \rvert < 1$ or $\lvert k \rvert > 1$, respectively. Also, flip it about the Y-axis, if k is negative (this latter statement follows from part ii)
(vii)	$y = f(x)$ to $y = \lvert f(x) \rvert$	: Reflect the parts of the graph that lie in the lower half (negative parts) into the upper half of the axes.
(viii)	$y = f(x)$ to $y = f(\lvert x \rvert)$	: Discard the left part of the graph (for $x < 0$) and take a reflection of this right part of the graph into the left half of the axes.
(ix)	$y = f(x)$ to $\lvert y \rvert = f(x)$	: Discard the lower part of the graph ($f(x) < 0$) and take a reflection of the upper part of the graph into the lower half of the axes.

MASTER STROKES

1. The domain of the definition of the function $f(x) = \sqrt{1 - \sqrt{1 - \sqrt{1 - x^2}}}$, is

(a) $(-\infty, 1)$ (b) $(-1, \infty)$
(c) $[0, 1]$ (d) $[-1, 1]$

2. The range of the function $f(x) = \log_e \sqrt{4 - x^2}$ is given by

(a) $(0, \infty)$ (b) $(-\infty, \infty)$
(c) $(-\infty, \log_e 2]$ (d) $(\log_e 2, \infty)$

3. The number of integral solutions of the equation $[x + 1] + 2x = 4[x + 1] - 6$, is

(a) 0 (b) 1 (c) 2 (d) 3

4. The domain of the definition of the function $f(x) = \log_4 [\log_5 \{\log_3 (18x - x^2 - 77)\}]$ is

(a) $(8, 10)$ (b) $[8, 10]$
(c) $(-\infty, 8)$ (d) $[10, \infty)$

5. The period of the function $f(x) = ||\sin x| - |\cos x||$, is

(a) $\frac{\pi}{2}$ (b) π
(c) 2π (d) None of these

6. The function $f(x)$ given by

$$f(x) = \begin{cases} x^4 \tan \frac{\pi x}{2}, & |x| < 1 \\ x|x|, & |x| \geq 1 \end{cases}$$ is

(a) an odd function
(b) an even function
(c) a periodic function
(d) None of the above

7. Let $f(x) = \log_e x$ and $g(x) = \frac{x^4 - 2x^3 + 3x^2 - 2x + 2}{2x^2 - 2x + 1}$

Then, the domain of $fog(x)$ is

(a) R (b) $[0, \infty)$
(c) $(0, \infty)$ (d) $[1, \infty)$

8. The function

$$f(x) = \cot^{-1} \sqrt{x(x + 3)} + \cos^{-1} \sqrt{x^2 + 3x + 1}$$

is defined on the set S, where S is equal to

(a) $\{-3, 0\}$ (b) $[-3, 0]$
(c) $[0, 3]$ (d) ϕ

9. The range of the function $f(x) = \sqrt{2 - x} + \sqrt{1 + x}$, is

(a) $[0, \sqrt{6}]$ (b) $[\sqrt{3}, \sqrt{6}]$
(c) $(\sqrt{3}, \sqrt{6})$ (d) $(0, \sqrt{3})$

10. If the function $f(x) = \lambda |\sin x| + \lambda^2 |\cos x| + \phi(\lambda)$ has period $\frac{\pi}{2}$, then λ is equal to

(a) 3 (b) 2 (c) 1 (d) 0

11. If $f : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to R$ given by $f(x) = \cos x + \sin\left[\frac{x + 1}{\lambda}\right]$ is an even function. Then, the set of values of λ $(\lambda > 0)$ is (Here, $[\cdot]$ denotes the greatest integer function)

(a) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) - \{0\}$ (b) $\left(\frac{\pi + 2}{2}, \infty\right)$
(c) $\left(0, \frac{\pi + 2}{2}\right) \cup \left(\frac{\pi + 2}{2}, \infty\right)$ (d) $\left(\frac{-\pi}{2}, \frac{\pi + 2}{2}\right) - \{0\}$

12. The domain of the function $f(x) = \sin^{-1}\left(\frac{8 \cdot 3^{x-2}}{1 - 3^{2(x-1)}}\right)$ is

(a) $(-\infty, 0]$ (b) $[2, \infty)$
(c) $(-\infty, 0] \cup [2, \infty)$ (d) $(-\infty, -1] \cup [1, \infty)$

13. Let $f(x) = (1 - x)^2 \sin^2 x + x^2$, $\forall\, x \in R$.

Consider the statements

P : There exists some $x \in R$ such that

$$f(x) + 2x = 2(1 + x^2)$$

Q : There exists some $x \in R$ such that

$$2f(x) + 1 = 2x(1 + x)$$

Then,

(a) both P and Q are true (b) P is true and Q is false
(c) P is false and Q is true (d) both P and Q are false

14. Let $f : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to R$ be given by

$f(x) = \{\log(\sec x + \tan x)\}^3$. Then, which is incorrect?

(a) $f(x)$ is an odd function (b) $f(x)$ is an one-one function
(c) $f(x)$ is an onto function (d) $f(x)$ is an even function

15. Let $f(x) = \sin\left\{\frac{\pi}{6} \sin\left(\frac{\pi}{2} \sin x\right)\right\}$, $\forall\ x \in R$ and $g(x) = \frac{\pi}{2} \sin x$, $\forall\, x \in R$. Let $fog(x)$ denotes $f\{g(x)\}$ and $gof(x)$ denotes $g\{f(x)\}$.

Then, which of the following is false?

(a) Range of f is $\left[-\frac{1}{2}, \frac{1}{2}\right]$

(b) Range of fog is $\left[-\frac{1}{2}, \frac{1}{2}\right]$

(c) $\lim_{x \to 0} \frac{f(x)}{g(x)} = \frac{\pi}{6}$

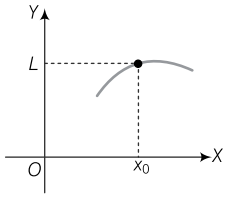
(d) There is an $x \in R$ such that $(gof)(x) = 1$

Answers

1. (d)	2. (c)	3. (b)	4. (a)	5. (a)
6. (a)	7. (a)	8. (a)	9. (b)	10. (c)
11. (b)	12. (c)	13. (c)	14. (d)	15. (d)

LIMITS

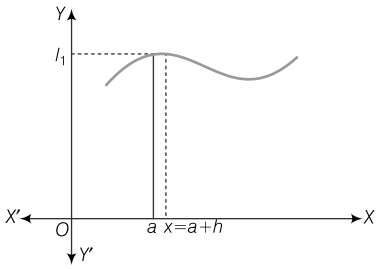
Let $f(x)$ be defined on an open interval about x_0, except possible at x_0 itself. If $f(x)$ gets arbitrarily close to L for all x sufficiently close to x_0, we say that f approaches the limit L as x approaches x_0 and we write $\lim_{x \to x_0} f(x) = L$.



Right Hand Limit

The right hand limit of $f(x)$ as x tends to a exist and is equal to l_1, if x approaches a through values greater than a, then $f(x)$ approaches a definite unique real number l_1,

i.e. $\lim_{x \to a^+} f(x) = l_1$



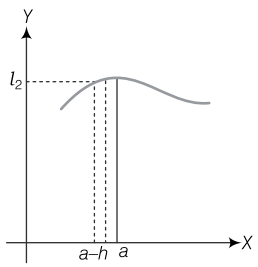
Note **Evaluate RHL** To evaluate $\lim_{x \to a^+} f(x)$

- Put $x = a + h$ in $f(x)$, then $\lim_{x \to a^+} f(x)$ reduces to $\lim_{h \to 0} f(a + h)$
- Simplify the limit as $h \to 0$

Left Hand Limit

The left hand limit as $x \to a$ exist and is equal to l_2, if x approaches a through values less than a, then $f(x)$ approaches a definite unique real number l_2.

i.e. $\lim_{x \to a^-} f(x) = l_2$



Note **Evaluate LHL** To evaluate $\lim_{x \to a} f(x)$

- Put $x = a - h$ in $f(x)$, then $\lim_{x \to a^-} f(x)$ reduces to $\lim_{h \to 0} f(a - h)$
- Simplify the limit as $h \to 0$

Existence of Limit

Limit to a function $f(x)$ exists, if $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ exist and both are equal.

Thus, $\lim_{x \to a} f(x)$ exists $\Leftrightarrow \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$

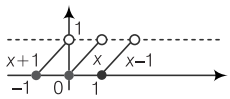
Reasons for Non-existence of the Limit

Here, student should aware about reasons for non-existence of limit. From the discussion, so far it is clear that $\lim_{x \to a} f(x)$ will not exist due to any of these three reasons.

(i) $f(x)$ is not defined in the neighbourhood of $x = a$.
e.g. Consider, $\lim_{x \to \pi/2} \sec^{-1}(\sin x)$.

(ii) $f(x)$ does not have a unique tendency.
e.g. Consider, $\lim_{x \to \infty} \cos\left(\frac{1}{x}\right)$.

(iii) Left and right tendencies of $f(x)$ are not same.
e.g. Consider, $\lim_{x \to 1} \{x\}$, where $\{\}$ denotes fractional part.



Note For the existence of the limit at $x = a$, $f(x)$ need not be defined at $x = a$. However, if $f(a)$ exists limit need not exist or even, if it exists, then its limit need not be equal to $f(a)$.

KEY POINTS

(i) If $\lim_{x \to a} f(x)g(x)$ exists, then we have following cases
(a) Both $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist.
(b) $\lim_{x \to a} f(x)$ exists and $\lim_{x \to a} g(x)$ does not exist.
(c) Both $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ do not exist.

(ii) If $\lim_{x \to a} [f(x) + g(x)]$ exists, then we can have the following cases
(a) If $\lim_{x \to a} f(x)$ exists, then $\lim_{x \to a} g(x)$ must exist.
(b) Both $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ do not exist.

Algebra of Limits

Let $\lim_{x\to a} f(x) = l$ and $\lim_{x\to a} g(x) = m$. If l and m exist, then

(i) $\lim_{x\to a}(f \pm g)(x) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = l \pm m$

(ii) $\lim_{x\to a}(fg)(x) = \lim_{x\to a} f(x) \lim_{x\to a} g(x) = lm$

(iii) $\lim_{x\to a}\left(\frac{f}{g}\right)(x) = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{l}{m}$, provided $m \neq 0$

(iv) $\lim_{x\to a} kf(x) = k \cdot \lim_{x\to a} f(x)$, where k is constant.

(v) $\lim_{x\to a}|f(x)| = |\lim_{x\to a} f(x)| = |l|$

(vi) $\lim_{x\to a}\{f(x)\}^{g(x)} = \lim_{x\to a} f(x)^{\lim_{x\to a} g(x)} = l^m$

(vii) $\lim_{x\to a} fog(x) = f\{\lim_{x\to a} g(x)\} = f(m)$, only if f is continuous at $g(x) = m$.

In particular,

(a) $\lim_{x\to a} \log f(x) = \log\{\lim_{x\to a} f(x)\} = \log l$

(b) $\lim_{x\to a} e^{f(x)} = e^{\lim_{x\to a} f(x)} = e^l$

Indeterminate Forms

In finding the limit of a function $f(x)$ at $x = a$, if a unique value cannot be assigned to $f(a)$; then $f(x)$ is said to be indeterminate at $x = a$.

A list of indeterminate forms is given below

(i) $\frac{\infty}{\infty} = \frac{1/\infty}{1/\infty} = \frac{0}{0}$, $0 \times \infty = \frac{0}{1/\infty} = \frac{0}{0}$

(ii) $y = 0^0 \Rightarrow \log y = \log(0^0) = 0 \times \log(0) = 0 \times \infty$

(iii) $y = \infty^0 \Rightarrow \log y = \log(\infty^0) = 0 \times \log(\infty) = 0 \times \infty$

(iv) $y = 1^\infty \Rightarrow \log y = \log(1^\infty) = \infty \times \log(1) = \infty \times 0$

(v) $\infty - \infty$ is also an indeterminate form as the ∞ and ∞ does not necessarily approach to the same infinity.

Evaluation of Limits

Evaluation of Algebraic Limits

(i) $\frac{0}{0}$ Form

(a) **Factorisation method**

Consider, $\lim_{x\to a} \frac{f(x)}{g(x)}$

If by substituting $x = a$, $\frac{f(x)}{g(x)}$ reduces to the form $\frac{0}{0}$.

Then, we first factorise the $f(x)$ and $g(x)$, then cancel out the common factor to evaluate the limit.

(b) **Rationalisation method** Rationalisation is followed when we have fractional powers (like 1/2, 1/3 etc) on expression in numerator or denominator or in both. After rationalisation the terms are factorised which on cancellation gives the result.

(c) **Based on standard formula** To evaluate the limit of the type $\lim_{x\to a} \frac{x^n - a^n}{x - a}$ we use following formulae

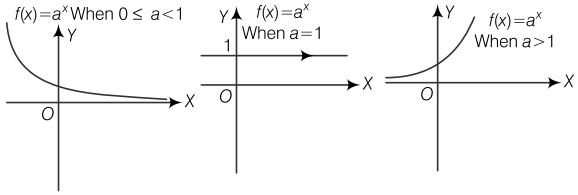
- $\lim_{x\to a} \frac{x^n - a^n}{x - a} = na^{n-1}$,

  where n is a rational number.
- $\lim_{x\to a} \frac{x^m - a^m}{x^n - a^n} = \frac{m}{n} a^{m-n}$,

  where m and n are rational numbers.

(ii) Algebraic Function of ∞ Type

(a) $\frac{\infty}{\infty}$ **form** Consider, the function a^x. First we should know the limiting values of $a^x (a > 0)$ as $X \to \infty$. See the graphs of these functions.



Now, see the graph for a^x when $a > 1$. This graph appears to touch X-axis in the negative side of X-axis and thereafter it increases rapidly.

That is why because $\lim_{x\to\infty} a^x \to 0$, again you will also find the result, $\lim_{x\to\infty} a^{x^{x\to\infty}} \to \infty$

Thus, we have $\lim_{x\to\infty} a^x = \begin{cases} \infty, & \text{if} \quad a > 1 \\ 1, & \text{if} \quad a = 1 \\ 0, & \text{if} \quad 0 \le a < 1 \end{cases}$

Note This type of problem are solved by taking the highest power of the terms tending to infinity as common numerator and denominator. That is after they are cancelled and the rest output is the result.

(b) $\infty - \infty$ **form** Such problems are simplified (generally rationalised) first, thereafter they generally acquire $\frac{\infty}{\infty}$ form.

SPEED BOOSTER

If m, n are positive integers and $a_0 \cdot b_0 \neq 0$ are non-zero real numbers. Then,

$$\lim_{x\to\infty} \frac{a_0x^m + a_1x^{m-1} + \ldots + a_{m-1}x + a_m}{b_0x^n + b_1x^{n-1} + \ldots + b_{n-1}x + b_n} = \begin{cases} 0, & m < n \\ \frac{a_0}{b_0}, & m = n \\ \infty, & m > n \text{ when } a_0b_0 > 0 \\ -\infty, & m > n \text{ when } a_0b_0 < 0 \end{cases}$$

Evaluation of Trigonometric Limits

To evaluate limits involving trigonometrical functions we have to follow the following steps.

***Step* I** Reduce the given expression in terms of $\sin\theta$ and $\cos\theta$.

***Step* II** If the expression obtained in step I is sum or difference of two terms, then remove the positive or negative signs in between two terms.

***Step* III** Arrange the terms and use the standard result given below (which is applicable).

If $\lim_{x\to a} f(x) = 0$, then

(i) $\lim_{x\to a} \frac{\sin f(x)}{f(x)} = \lim_{x\to a} \frac{\tan f(x)}{f(x)} = 1$

(ii) $\lim_{x\to a} \cos f(x) = 1$

(iii) $\lim_{x\to a} \frac{\sin^{-1} f(x)}{f(x)} = \lim_{x\to a} \frac{\tan^{-1} f(x)}{f(x)} = 1$

Evaluation of Logarithmic Limits

In this section, we shall deal with the problems based on expansion of logarithmic which is given below.

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots \infty, \text{ where } -1 \le x \le 1$$

And it should be noted that expansion is true only, if the base is e. To evaluate the logarithmic limit, we use the following formula

$$\lim_{x\to 0} \frac{\log(1+x)}{x} = 1$$

Evaluation of Exponential Limits

(i) **Based on series expansion**

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \infty$$

To evaluate the exponential limit, we use the following results

(a) $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$

(b) $\lim_{x\to 0} \frac{a^x - 1}{x} = \log e^a$

(ii) **Evaluation of exponential limits of the form 1^∞**

To evaluate the exponential form 1^∞, we use the following results

If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$, then

$\lim_{x\to a} \{1 + f(x)\}^{1/g(x)} = e^{\lim_{x\to a} \frac{f(x)}{g(x)}}$

When $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} g(x) = \infty$

Then, $\lim_{x\to a} \{f(x)\}^{g(x)} = \lim_{x\to a} [1 + f(x) - 1]^{g(x)}$

$= e^{\lim_{x\to a} [f(x) - t]g(x)}$

POINTS TO REMEMBER

(i) $\lim_{x\to 0} (1+x)^{1/x} = e$ (ii) $\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x = e$

(iii) $\lim_{x\to 0} (1+\lambda x)^{1/x} = e^{\lambda}$ (iv) $\lim_{x\to\infty} \left(1 + \frac{\lambda}{x}\right)^{1/x} = e^{\lambda}$

Evaluation of Limits of the Form 0^0

When $\lim_{x\to a} f(x) \neq 1$ but $f(x)$ is positive in the neighbourhood of $x = a$

In this case, we write $\{f(x)\}^{g(x)} = e^{\log_e \{f(x)\}^{g(x)}}$

$\Rightarrow \lim_{x\to a} \{f(x)\}^{g(x)} = e^{\lim_{x\to a} g(x)\log_e f(x)}$

Some Standard Limits

(i) If $p(x)$ is a polynomial, then $\lim_{x\to a} p(x) = p(a)$

(ii) $\lim_{x\to 0} (1+x)^{1/x} = e$

(iii) $\lim_{x\to 0} \frac{(1+x)^n - 1}{x} = n$

(iv) $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$

(v) $\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1$

(vi) $\lim_{x\to 0} \frac{\sin^{-1} x}{x} = 1 = \lim_{x\to 0} \frac{\tan^{-1} x}{x}$

(vii) $\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\tan x}{x} = \lim_{x\to 0} \cos x = 1$

[where, x is in radians]

(viii) $\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x = e$

(ix) $\lim_{x\to 0} \frac{a^x - 1}{x} = \ln(a), a \in R^+$

(x) $\lim_{x\to a} \frac{x^m - a^m}{x^n - a^n} = \frac{m}{n} a^{m-n}$

(xi) $\lim_{x\to 0} \frac{\log_a(1+x)}{x} = \log_a e; a > 0, \neq 1$

If $\lim_{x\to a} f(x) = 0$, then the following results will be holding true :

(xii) $\lim_{x\to a} \frac{\sin f(x)}{f(x)} = \lim_{x\to a} \frac{\tan f(x)}{f(x)} = \lim_{x\to a} \cos f(x) = 1$

(xiii) $\lim_{x\to a} \frac{\sin^{-1} f(x)}{f(x)} = \lim_{x\to a} \frac{\tan^{-1} f(x)}{f(x)} = 1$

(xiv) $\lim_{x\to a} \frac{b^{f(x)} - 1}{f(x)} = \ln b \, (b > 0)$

(xv) $\lim_{x\to a} [1 + f(x)]^{1/f(x)} = e$

Summation of Series Using Definite Integral as the Limit

The expression of the form, $\lim_{x \to \infty} \frac{1}{n} \sum_{r=\phi(x)}^{\psi(x)} f\left(\frac{r}{n}\right) = \int_a^b f(x)dx$,

(i) Σ is replaced by $\int$

(ii) $\frac{r}{n}$ is replaced by x

(iii) $\frac{1}{n}$ is replaced by dx

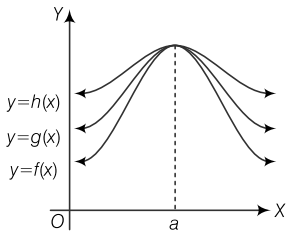
(iv) To obtain lower and upper limits, we have $a = \lim_{n \to \infty} \frac{\phi(x)}{n}$ and $b = \lim_{n \to \infty} \frac{\psi(x)}{n}$

The value, so obtained is the required sum of the given series.

Sandwich Theorem for Evaluating Limits

Let f, g and h be real functions such that $f(x) \le g(x) \le h(x)$ for all x in the common domain of definition. For some real number a, if $\lim_{x \to a} f(x) = l = \lim_{x \to a} h(x)$, then $\lim_{x \to a} g(x) = l$.

This is illustrated in figure.



Series Expansion

Following are some of the frequently used series expansions.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

$$a^x = 1 + (\ln a)x + (\ln a)^2 \frac{x^2}{2!} + \ldots, a \in R^+$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots, n \in R, |x| < 1$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots, -1 < x \le 1$$

L' HOSPITAL'S RULE

- If $\frac{f(x)}{g(x)}$ has an indeterminancy at $x = 0$ of the type $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$, where f' is called the derivative of f.
- If $\frac{f'(x)}{g'(x)}$ too has an indeterminancy at $x = a$ of the type $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then $\lim_{x \to a} \frac{f'(x)}{g'(x)} = \lim_{x \to a} \frac{f''(x)}{g''(x)}$
- This can be done continued till we find a determinate result.

MASTER STROKES

1. The value of $\lim_{x \to 2} \frac{5}{\sqrt{2} - \sqrt{x}}$ is

(a) $10\sqrt{2}$
(b) ∞
(c) $-\infty$
(d) non-existent

2. If $S_1 = \Sigma n$, $S_2 = \Sigma n^2$, $S_3 = \Sigma n^3$, then the value of $\lim_{n \to \infty} \frac{S_1\left(1 + \frac{S_3}{8}\right)}{S_2^2}$ is equal to

(a) $\frac{9}{64}$ (b) $\frac{9}{32}$
(c) $\frac{3}{64}$ (d) $\frac{3}{32}$

3. If $[\cdot]$ denotes the greatest integer function, then $\lim_{x \to \frac{\pi}{2}} \left[\frac{x - \frac{\pi}{2}}{\cos x}\right]$ is equal to

(a) 1 (b) 2 (c) –2 (d) –1

4. $\lim_{x \to 0} \frac{\cos(\sin x) - 1}{x^2}$ is equal to

(a) 1 (b) –1 / 2 (c) 1 / 2 (d) –1

5. The value of $\lim_{x \to 2} \frac{e^{3x-6} - 1}{\sin(2-x)}$, is

(a) $\frac{3}{2}$ (b) 3 (c) –3 (d) –1

6. $\lim_{x \to 0}(\cos x)^{\cot x}$ is equal to

(a) e (b) $\frac{1}{e}$ (c) 1 (d) -1

7. $\lim_{x \to \infty}\left(\frac{x^2+4x-3}{x^2-2x+5}\right)^x$ is equal to

(a) e^6 (b) e^{-6} (c) e^2 (d) e^4

8. If $\lim_{x \to 0}\frac{\cos 4x + a\cos 2x + b}{x^4}$ is finite, then (a, b) is equal to

(a) $(5, -4)$ (b) $(-5, -4)$
(c) $(-4, 3)$ (d) $(4, 5)$

9. The value of $\lim_{x \to 0}\frac{e^x + \log(1+x) - (1-x)^{-2}}{x^2}$ is equal to

(a) 0 (b) -3 (c) -1 (d) infinity

10. The integer n for which $\lim_{x \to 0}\frac{(\cos x - 1)(\cos x - e^x)}{x^n}$ is a finite non-zero number, is

(a) 1 (b) 2 (c) 3 (d) 4

11. $\lim_{x \to 0}\frac{(1-\cos 2x)(3+\cos x)}{x\tan 4x}$ is equal to

(a) $-1/4$ (b) $1/2$ (c) 1 (d) 2

12. If $f(x) = \lim_{n \to \infty}\sum_{r=0}^{n}\frac{\tan\left(\frac{x}{2^{r+1}}\right) + \tan^3\left(\frac{x}{2^{r+1}}\right)}{1 - \tan^2\left(\frac{x}{2^{r+1}}\right)}$, then $\lim_{x \to 0}\frac{f(x)}{x}$ is equal to

(a) 1 (b) 0
(c) -1 (d) None of these

13. If $\lim_{x \to \infty}(8x^3 + mx^2)^{1/3} - nx$ exists and is equal to 1, then the value of $\frac{m}{n}$ is

(a) $\frac{1}{6}$ (b) 6 (c) 3 (d) $\frac{1}{3}$

14. If A, B, C are positive real numbers such that $\lim_{x \to \infty}(\sqrt{Ax^2 + Bx} - Cx) = 2$, then $\frac{BC}{A}$ equals

(a) 4 (b) 2 (c) $\frac{1}{2}$ (d) 1

15. $\lim_{x \to 0}\frac{2^{|x|}e^{|x|} - |x|\log_2 2 - 1}{x\tan x}$ is equal to

(a) $\frac{1}{2}(\ln 2)^2 + \frac{1}{2}(\ln 2) + 1$ (b) $(\ln 2)^2 + \frac{1}{2}(\ln 2) + 1$
(c) $(\ln 2)^2 + (\ln 2) + \frac{1}{2}$ (d) $\frac{1}{2}(\ln 2)^2 + (\ln 2) + \frac{1}{2}$

16. The value of $\lim_{x \to 1}\left\{\frac{x^n - 1}{n(x-1)}\right\}^{\frac{1}{x-1}}$ is

(a) $e^{\frac{n}{2}}$ (b) $e^{\frac{n}{2}-1}$ (c) $e^{\frac{n-1}{2}}$ (d) $e^{\frac{n+1}{2}}$

17. The value of $\lim_{x \to 0}\frac{\sin(\sin x) - \tan(\sin x)}{\sin^3(\sin x)}$ is

(a) $\frac{-1}{2}$ (b) -1 (c) $\frac{1}{2}$ (d) 1

18. $\lim_{x \to \infty}\frac{x^{2^{32}} - 2^{32}x + 4^{16} - 1}{(x-1)^2}$ is equal to

(a) $2^{63} - 2^{31}$ (b) $2^{64} - 2^{31}$
(c) $2^{62} - 2^{31}$ (d) $2^{65} - 2^{31}$

19. Let m and n be two positive integers greater than 1. If $\lim_{\alpha \to 0}\left(\frac{e^{\cos(\alpha^n)} - e}{\alpha^m}\right) = -\frac{e}{2}$, then the value of $\frac{m}{n}$ is

(a) 2 (b) 4 (c) 6 (d) 9

20. $\lim_{x \to 0}\frac{\sin\{\pi(\cos^2 x)\}}{x^2}$ is equal to

(a) $-\pi$ (b) π (c) $\frac{\pi}{2}$ (d) 1

21. $\lim_{x \to 2}\frac{\sum_{r=1}^{n} x^r - \sum_{r=1}^{n} 2^r}{x-2}$ is equal to

(a) n (b) $(n-1)2^n$
(c) $(n-1)2^n + 1$ (d) None of these

22. The value of $\lim_{x \to \infty}\frac{2x^{1/2} + 3x^{1/3} + 4x^{1/4} + \ldots + nx^{1/n}}{(2x-3)^{1/2} + (2x-3)^{1/3} + \ldots + (2x-3)^{1/n}}$ is

(a) 0 (b) 2 (c) $\sqrt{2}$ (d) $\frac{1}{\sqrt{3}}$

23. Let $f(\theta) = \frac{1}{\tan^9\theta}\{(1+\tan\theta)^{10} + (2+\tan\theta)^{10} + \ldots + (20+\tan\theta)^{10}\} - 20\tan\theta$. The left hand limit of $f(\theta)$ as $\theta \to \frac{\pi}{2}$ is

(a) 1900 (b) 2000 (c) 2100 (d) 2200

24. Let the sequence $<b_n>$ of real numbers satisfy the reocurrence relation : $b_{n+1} = \frac{1}{3}\left(2b_n + \frac{125}{b_n^2}\right)$, $b_n \neq 0$, then $\lim_{n \to \infty} b_n$ is equal to

(a) 0 (b) ∞ (c) 5 (d) $\frac{2}{3}$

25. The value of $\lim_{n \to \infty}\{\sqrt[3]{n^2 - n^3} + n\}$, is

(a) $\frac{1}{3}$ (b) $\frac{-1}{3}$ (c) $\frac{2}{3}$ (d) $\frac{-2}{3}$

Answers

1. (d)	2. (a)	3. (c)	4. (b)	5. (c)
6. (c)	7. (a)	8. (c)	9. (b)	10. (c)
11. (d)	12. (a)	13. (b)	14. (a)	15. (d)
16. (c)	17. (a)	18. (a)	19. (a)	20. (b)
21. (c)	22. (a)	23. (c)	24. (c)	25. (a)

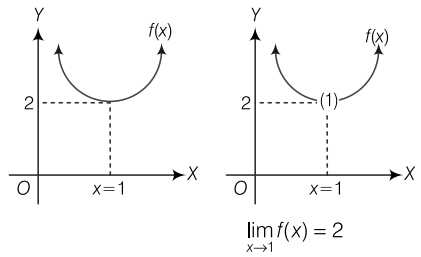
CONTINUITY AND DIFFERENTIABILITY

Continuity

A continuous function is a function, whose graph can be drawn without lifting the pen from the paper. Graphically, it could be stated as shown in figure (i).

When the function $f(x)$ is continuous at a point $x = a$, we mean that at point $(a, f(a))$ the graph of the function has no hole or gaps. Here, $\lim_{x \to 1} f(x) = 2$ and $f(1) = 2$.

Thus, $\lim_{x \to 1} f(x) = f(1)$, hence $f(x)$ is continuous.



(i) (ii)

A function $f(x)$ is said to be continuous at $x = a$, if $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a)$.

Continuity in Interval

(i) **Continuity in an open interval**

A function $f(x)$ is said to be continuous in an open interval (a, b), if it is continuous at each point of (a, b).

(ii) **Continuity in a closed interval**

A function $f(x)$ is said to be continuous in a closed interval $[a, b]$, if

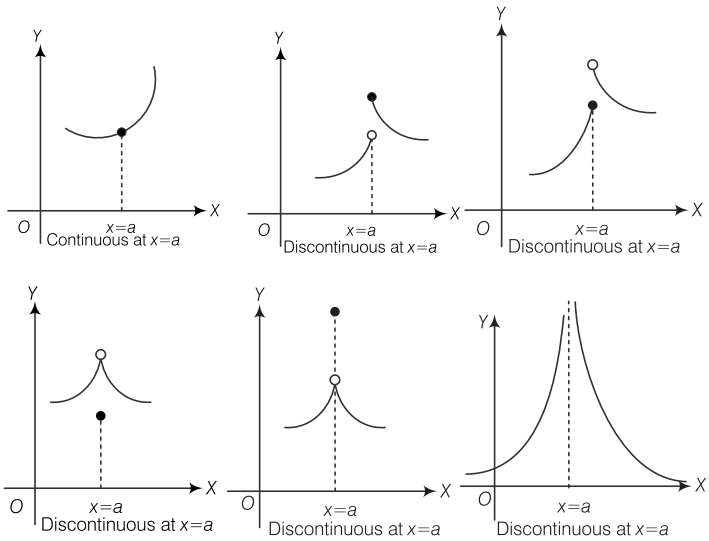
(a) $f(x)$ is continuous from right at $x = a$
i.e. $\lim_{x \to a^+} f(x) = f(a)$

(b) $f(x)$ is continuous from left at $x = b$
i.e. $\lim_{x \to b^-} f(x) = f(b)$

(c) $f(x)$ is continuous at each point of the interval (a, b).

Discontinuity
(Graphical Approach)

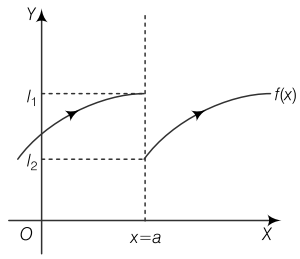
In case one has to lift his pen, when moving through the graph of a function, is said to have a break or discontinuous at $x = a$. Different types of situation, which may come up at $x = a$ along the graph, can be shown below.



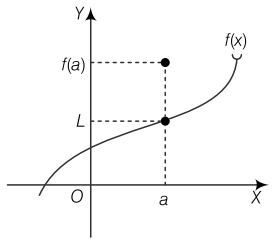
Discontinuity of a Function

If $f(x)$ is not continuous at $x = a$, then we say that $f(x)$ is discontinuous at $x = a$. $f(x)$ will be discontinuous at $x = a$ in any of the following cases.

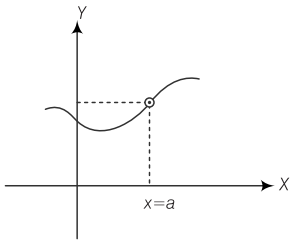
(i) $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ exist and are not equal.



(ii) $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ exist and are equal but not equal to $f(a)$.



(iii) $f(a)$ is not defined.



Theorems on Continuity

(i) Sum, difference, product and quotient of two continuous functions are always continuous function. However, $h(x) = \frac{f(x)}{g(x)}$ is continuous at $x = a$ only if $g(a) \neq 0$.

(ii) If $f(x)$ is continuous and $g(x)$ is discontinuous at $x = a$, then product function $\phi(x) = f(x) \cdot g(x)$ is not necessarily be discontinuous at $x = a$.

(iii) If $f(x)$ and $g(x)$ both are discontinuous at $x = a$, then the product function $\phi(x) = f(x)g(x)$ is not necessarily be discontinuous at $x = a$.

Continuity of Composite Function

If the function $u = f(x)$ is continuous at the point $x = a$ and the function $y = g(u)$ is continuous at the point $x = f(a)$, then the composite function $y = gof(x) = g(f(x))$ is continuous at that point.

Note

- If f is continuous on a closed interval $[a, b]$, then it is bounded.
- A continuous function whose domain is some closed interval must have its range also in the closed interval.
- If $f(a)$ and $f(b)$ possess opposite signs, then there exists atleast one solution of the equation $f(x) = 0$ in the open interval (a, b) provided f is continuous in $[a, b]$.
- If f is continuous on $[a, b]$, then f^{-1} is also continuous.

Best Practice SHOTS

1. The function $f : R - \{0\} \to R$ given by $f(x) = \frac{1}{x} - \frac{2}{e^{2x} - 1}$ can be made continuous at $x = 0$ by defining $f(0)$ as

(a) 0 (b) 1
(c) 2 (d) – 1

2. If $f(x) = \begin{cases} \frac{(4^x - 1)^3}{\sin\left(\frac{x}{4}\right) \log\left(1 + \frac{x^2}{3}\right)} & , x \neq 0 \\ k & , x = 0 \end{cases}$

is continuous at $x = 0$, then k is equal to

(a) $12 (\log_e 4)^2$
(b) $96 (\log_e 2)^3$
(c) $(\log_e 4)^3$
(d) None of the above

3. If $f(x) = \begin{cases} \frac{A + 3\cos x}{x^2} & , x < 0 \\ B \tan \frac{\pi}{[x + 3]} & , x \geq 0 \end{cases}$

Where $[\cdot]$ represents the greatest integer function, is continuous at $x = 0$. Then,

(a) $A = -3, B = -\sqrt{3}$
(b) $A = 3, B = \frac{-\sqrt{3}}{2}$
(c) $A = -3, B = \frac{-\sqrt{3}}{2}$
(d) $A = \frac{-\sqrt{3}}{2}, B = -3$

4. The function $f(x) = x - [x]$, where $[\cdot]$ denotes the greatest integer function, is

(a) continuous everywhere
(b) continuous at integer point only
(c) continuous at non-integer points only
(d) nowhere continuous

5. Let $f(x)$ be given by $f(x) = \begin{cases} x & , \text{ if } x \\ 1 - x & , \text{ if } x \end{cases}$

The number of points at which $f(x)$ is continuous, is

(a) ∞ (b) 1
(c) 0 (d) None of these

6. The set of points of discontinuity of the function $f(x) = \frac{1}{\log|x|}$ is

(a) $\{0\}$ (b) $\{-1, 1\}$
(c) $\{-1, 0, 1\}$ (d) None of these

7. Let $f(x) =$ degree of $(u^{x^2} + u^2 + 2u + 3)$. Then, at $x = \sqrt{2}$, $f(x)$ is

(a) continuous
(b) discontinuous
(c) Both (a) and (b)
(d) None of the above

Answers

1. (b) **2.** (b) **3.** (c) **4.** (c) **5.** (c)
6. (c) **7.** (a)

DIFFERENTIABILITY

Let $f(x)$ be a real valued function defined on an open interval (a, b), where $c \in (a, b)$.Then, $f(x)$ is said to be differentiable or derivable at $x = c$, if and only if $\lim_{x \to c} \frac{f(x) - f(c)}{(x - c)}$ exists finitely.

This limit is called the derivative or differentiable coefficient of the function $f(x)$ at $x = c$ and is denoted by $f'(c)$ or $Df(c)$ or $\frac{d}{dx}(f(x))_{x=c}$

$$\lim_{x \to c^-} \frac{f(x) - f(c)}{(x - c)} = \lim_{h \to 0} \frac{f(c - h) - f(c)}{-h}$$

is called the left hand derivative of $f(x)$ at $x = c$ and is denoted by $f'(c^-)$ or $Lf'(c)$.

While $\lim_{x \to c^+} \frac{f(x) - f(c)}{(x - c)} = \lim_{h \to 0} \frac{f(c + h) - f(c)}{h}$ is called the right hand derivative of $f(x)$ at $x = c$ and is denoted by $f'(c^+)$ or $Rf'(c)$.

If $Lf'(c) \neq Rf'(c)$, we say that $f(x)$ is not differentiable at $x = c$.

Differentiability in a Set

(i) A function $f(x)$ defined on an open interval (a, b) is said to be differentiable or derivable in open interval (a, b), if it is differentiable at each point of (a, b).

(ii) A function $f(x)$ defined on $[a, b]$ is said to be differentiable or derivable at the end points a and b, if it is differentiable from the right at a and from the left at b. In other words,

$\lim_{x \to a^+} \frac{f(x) - f(a)}{x - a}$ and $\lim_{x \to b^-} \frac{f(x) - f(b)}{x - b}$ both exist.

If f is derivable in the open interval (a, b) and also the end points a and b, then f is said to be derivable in the closed interval $[a, b]$.

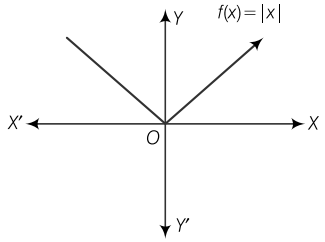
Note A function f is said to be differentiable function, if it is differentiable at every point of its domain.

Relation between Continuity and Differentiability

If a function is differentiable at a point, then it is necessarily continuous at that point. But converse is not necessarily true or $f(x)$ is differentiable at $x = c \Rightarrow f(x)$ is continuous at $x = c$ but converse is not necessarily true.

e.g. The function $f(x) = |x|$ is continuous at $x = 0$ but it is not differentiable at $x = 0$, as shown in figure given below.

Which shows we have sharp edge at $x = 0$. Hence, the function is not differentiable but continuous at $x = 0$.



Note

If $f(x)$ is differentiable, then its graph must be smooth, i.e. there should be no break or corner. Thus, for a function $f(x)$

- Differentiable $\Rightarrow$ Continuous
- Continuous $\Rightarrow$ May or may not be differentiable
- Not continuous $\Rightarrow$ Not differentiable

Theorems on Differentiability

(i) If $f(x)$ and $g(x)$ are both derivable at $x = a$, then $f(x) \pm g(x)$, $f(x) \cdot g(x)$ and $\frac{f(x)}{g(x)}$ are also derivable at $x = a$, $\left[\text{only if } g(a) \neq 0 \text{ for } \frac{f(x)}{g(x)}\right]$

(ii) If $f(x)$ is derivable at $x = a$ and $g(x)$ is not differentiable at $x = a$, then $f(x) \pm g(x)$ will not be derivable at $x = a$.

(iii) If both $f(x)$ and $g(x)$ are non-derivable, then nothing can be said about sum/difference/product functions.

(iv) If $f(x)$ is derivable at $x = a$ and $f(a) = 0$ and $g(x)$ is continuous at $x = a$. Then, the product function $h(x) = f(x) \cdot g(x)$ will be derivable at $x = a$.

Note If $y = f(x)$ is differentiable at $x = a$, then it is not necessarily that the derivative is continuous at $x = a$.

Some Standard Differentiable Function

(i) Exponential function a^x, $a > 0$ is differentiable at each $x \in R$.

(ii) The logarithmic function is differentiable at each point in its domain.

(iii) Absolute value functions are always continuous throughout but not differentiable at their critical points.
e.g. $|x - a|$ is continuous but not differentiable at $x = a$.

(iv) Every polynomial function is differentiable at each $x \in R$.

(v) Every constant function is differentiable at each $x \in R$.

(vi) Trigonometric and inverse trigonometric functions are differentiable in their respective domain.

(vii) The composition of differentiable functions is differentiable.

Best Practice SHOTS

8. It is given that $f'(a)$ exists, then $\lim_{x \to a} \frac{x f(a) - a f(x)}{x - a}$ is equal to

(a) $f(a) - a f'(a)$ (b) $f'(a)$
(c) $-f'(a)$ (d) $f(a) + a f'(a)$

9. If $f(4) = 4, f'(4) = 1$, then $\lim_{x \to 4} \frac{2 - \sqrt{f(x)}}{2 - \sqrt{x}}$ is equal to

(a) -1 (b) 1 (c) 2 (d) -2

10. Let $f : R \to R$ be a function defined by $f(x) = \min \{x + 1, |x| + 1\}$. Then, which of the following is true?

(a) $f(x) \geq 1, \forall x \in R$
(b) $f(x)$ is not differentiable at $x = 1$
(c) $f(x)$ is everywhere differentiable
(d) $f(x)$ is not differentiable at $x = 0$

11. Let $f(x) = \begin{cases} (x-1)\sin\left(\frac{1}{x-1}\right), & \text{if } x \neq 1 \\ 0, & \text{if } x = 1 \end{cases}$

Then, which one of the following is true?

(a) f is differentiable at $x = 0$ but not at $x = 1$
(b) f is differentiable at $x = 1$ but not at $x = 0$
(c) f is neither differentiable at $x = 0$ nor at $x = 1$
(d) f is differentiable at $x = 0$ and at $x = 1$

12. Let $g(x) = \frac{(x-1)^n}{\log\{\cos^m(x-1)\}}; 0 < x < 2,$

m and n are integers, $m \neq 0, n > 0$ and let p the left hand derivative of $|x - 1|$ at $x = 1$. If $\lim_{x \to 1^+} g(x) = p$, then

(a) $n = 1, m = -2$
(b) $n = 1, m = -1$
(c) $n = 2, m = 2$
(d) $n > 2, m = n$

Answers

8. (a) **9.** (b) **10.** (c) **11.** (a) **12.** (c)

MASTER STROKES

1. The function $f(x) = [x]^2 - [x^2]$ (where $[\cdot]$ denotes the greatest integer function) is discontinuous at

(a) all integers
(b) all integers except 0 and 1
(c) all integers except 0
(d) all integers except 1

2. Let $f(x) = \begin{cases} |x| - 3, & x < 1 \\ |x - 2| + a, & x \geq 1 \end{cases}$

$g(x) = \begin{cases} 2 - |x|, & x < 1 \\ \text{sgn}(x) - b, & x \geq 1 \end{cases}$

If $h(x) = f(x) + g(x)$ is discontinuous at exactly one point, then which of the following are correct?

(a) $a = -3, b = 0$
(b) $a = -3, b = -1$
(c) $a = 2, b = 0$
(d) $a = 0, b = 3$

3. The value of k for which

$$f(x) = \begin{cases} x\left(1 + x e^{-\frac{1}{x^2}} \sin\left(\frac{1}{x^4}\right)\right)^{e^{\frac{1}{x^2}}}, & x \neq 0 \\ k, & x = 0 \end{cases}$$

is continuous at $x = 0$, is

(a) 1 (b) 2
(c) 3 (d) 4

4. Let $f(x) = \begin{cases} 3\sin x + a^2 - 10a + 30, & x \notin Q \\ 4\cos x, & x \in Q \end{cases}$. Which one of the following statements is correct?

(a) $f(x)$ is continuous for all x, when $a = 5$
(b) $f(x)$ must be continuous for all x, when $a = 5$
(c) $f(x)$ is continuous for all $x = 2n\pi - \tan^{-1}\frac{4}{3}, n \in Z$, when $a = 5$
(d) $f(x)$ is continuous for all $x = 2n\pi - \tan^{-}\frac{4}{3}, n \in Z$, when $a = 5$

5. Let a and b be real numbers such that the function $g(x) = \begin{cases} -3ax^2 - 2, & x < 1 \\ bx + a^2, & x \geq 1 \end{cases}$ is differentiable for all $x \in R$. Then, the possible values of a are

(a) 1, 2 (b) 3, 4
(c) 5, 6 (d) 8, 9

6. Let $f : R \to R$ and $g : R \to R$ be respectively given by $f(x) = |x| + 1$ and $g(x) = x^2 + 1$. Define $h : R \to R$ by

$h(x) = \begin{cases} \max\{f(x), g(x)\}, & \text{if } x \leq 0 \\ \min\{f(x), g(x)\}, & \text{if } x > 0 \end{cases}$

The number of points at which $h(x)$ is not differentiable is

(a) 1 (b) 2
(c) 3 (d) 4

7. For every integer n, let a_n and b_n be real numbers. Let function $f: R \to R$ be given by

$$f(x) = \begin{cases} a_n + \sin \pi x & \text{for } x \in [2n, 2n+1] \\ b_b + \cos \pi x & \text{for } x \in (2n-1, 2n) \end{cases}$$

for all integers n. If f is continuous, then which of the following does not hold for all n?

(a) $a_n - b_{n+1} = -1$
(b) $a_{n-1} - b_{n-1} = 0$
(c) $a_n - b_n = 1$
(d) $a_{n-1} - b_n = -1$

8. Let $f: R \to R$ be a function such that $f(x+y) = f(x) + f(y), \forall\, x, y \in R$

If $f(x)$ is differentiable at $x = 0$, then which one of the following is incorrect?

(a) $f(x)$ is continuous, $\forall\, x \in R$
(b) $f'(x)$ is constant, $\forall\, x \in R$
(c) $f'(x)$ is differentiable, $\forall\, x \in R$
(d) $f'(x)$ is differentiable only in a finite interval containing zero

9. Let $f(x) = \begin{cases} x^2 \left|\cos \dfrac{\pi}{x}\right| & , \quad x \neq 0 \\ 0 & , \quad x = 0 \end{cases}$. Then, f is

(a) differentiable both at $x = 0$ and $x = 2$
(b) differentiable at $x = 0$ but not differentiable at $x = 2$
(c) not differentiable at $x = 0$ but differentiable at $x = 2$
(d) differentiable neither at $x = 0$ nor at $x = 2$

10. If $f(x) = \begin{cases} -x - \dfrac{\pi}{2} & , \quad x \leq -\dfrac{\pi}{2} \\ -\cos x & , \quad -\dfrac{\pi}{2} < x \leq 0, \\ x - 1 & , \quad 0 < x \leq 1 \\ x & , \quad x > 1 \end{cases}$

then which one of the following is not correct?

(a) $f(x)$ is continuous at $x = -\dfrac{\pi}{2}$
(b) $f(x)$ is not differentiable at $x = 0$
(c) $f(x)$ is differentiable at $x = 1, -\dfrac{3}{2}$
(d) $f(x)$ is discontinuous at $x = 0$

11. If $f(x) = \min\{1, x^2, x^3\}$. Then,

(a) $f(x)$ is everywhere continuous
(b) $f(x)$ is continuous and differentiable everywhere
(c) $f(x)$ is not differentiable at two points
(d) $f(x)$ is differentiable at one point only

12. The function $f(x) = ||x| - 1|$, $x \in R$ is differentiable at all $x \in R$ except at the points

(a) $1, 0, -1$ (b) 1 (c) $1, -1$ (d) -1

13. If $f(x) = [\sin^2 x]$ ($[\cdot]$ denotes the greatest integer function), then

(a) f is everywhere continuous
(b) f is everywhere differentiable
(c) f is a constant function
(d) None of the above

14. If $f(x) = \dfrac{1}{1-x}$ and $f^n = fofofo....of$, then the points of discontinuity of $f^{3n}(x)$ is/are

(a) $x = 2$ (b) $x = 0, 1$
(c) $x = 1, 2$ (d) $x = 1$

15. If the function $f(x)$ given by

$$f(x) = \begin{cases} 2^{\frac{1}{x-1}} & , \quad x < 1 \\ ax^2 + bx & , \quad x \geq 1 \end{cases}$$

is everywhere differentiable, then

(a) $a = 0, b = 1$ (b) $a = 0, b = 0$
(c) $a = 1, b = 0$ (d) None of these

Answers

1. (d)	**2.** (b)	**3.** (a)	**4.** (c)	**5.** (a)
6. (c)	**7.** (b)	**8.** (d)	**9.** (b)	**10.** (d)
11. (a)	**12.** (a)	**13.** (d)	**14.** (b)	**15.** (b)

DIFFERENTIATION

The rate of change of quantity 'y' with respect to another quantity x is called derivative or differential coefficient of y with respect to x and is denoted by $\dfrac{dy}{dx}$.

Rules for Differentiation

(i) If $f(x)$ and $g(x)$ are differentiable functions and $\alpha \in R$, then $\dfrac{d}{dx}[f(x) \pm g(x)] = \dfrac{d}{dx}[f(x)] \pm \dfrac{d}{dx}[g(x)]$

and $\dfrac{d}{dx}[\alpha f(x)] = \alpha \dfrac{d}{dx} f(x)$.

(ii) **Product rule** If $f(x)$ and $g(x)$ are differentiable functions, then

$$\frac{d}{dx}[f(x) \cdot g(x)] = g(x)\frac{d}{dx}[f(x)] + f(x)\frac{d}{dx}[g(x)]$$

SPEED BOOSTER

$$d/dx\{f(x) \cdot g(x) \cdot h(x)\} = f(x) \cdot g(x) \cdot h'(x) + f(x) \cdot g'(x) \cdot h(x) + f'(x) \cdot g(x) \cdot h(x)$$
$$= \frac{(fg)'h + (gh)'f + (hf)'g}{2}$$

(iii) **Quotient rule**

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{g(x)\frac{d}{dx}[f(x)]-f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

(iv) **Chain rule** If $y=h(u)$ and $u=f(x)$, then

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$$

This rule can be extended as follows

If $y=u(x), u=v(x), v=f(x)$ then

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dv}\cdot\frac{dv}{dx}$$

Note If at all points of a certain interval, $f'(x)=0$, then the function $f(x)$ preserves a constant value within this interval. This function is said to be constant function.

Derivative of Some Standard Functions

(i) The trigonometric functions have the following derivatives

$\frac{d}{dx}(\sin x)=\cos x$, $\frac{d}{dx}(\cos x)=-\sin x$

$\frac{d}{dx}(\tan x)=\sec^2 x$, $\frac{d}{dx}(\cot x)=-\operatorname{cosec}^2 x$

$\frac{d}{dx}(\sec x)=\sec x\tan x$, $\frac{d}{dx}(\operatorname{cosec} x)=-\operatorname{cosec} x\cot x$

(ii) $\frac{d}{dx}(x^n)=nx^{n-1}$, if $f(x)$ is a differentiable function then using chain rule $\frac{d}{dx}[f(x)^n]=n[f(x)]^{n-1}f'(x)$.

(iii) $\frac{d}{dx}(\log x)=\frac{1}{x}$, if $f(x)$ is a differentiable function then $\frac{d}{dx}[\log f(x)]=\frac{1}{f(x)}f'(x)$.

(iv) $\frac{d}{dx}(a^x)=a^x\log a$.

In particular, $\frac{d}{dx}(e^x)=e^x$ and $\frac{d}{dx}(e^{f(x)})=e^{f(x)}f'(x)$.

(v) $\frac{d}{dx}(\sin^{-1}x)=-\frac{d}{dx}(\cos^{-1}x)=\frac{1}{\sqrt{1-x^2}}$ for $-1<x<1$.

At the points $x=\pm1$, $\sin^{-1}x$ and $\cos^{-1}x$ are not differentiable.

(vi) $\frac{d}{dx}(\tan^{-1}x)=-\frac{d}{dx}(\cot^{-1}x)=\frac{1}{1+x^2}$ for all $x\in R$.

(vii) $\frac{d}{dx}(\sec^{-1}x)=-\frac{d}{dx}(\operatorname{cosec}^{-1}x)=\frac{1}{|x|\sqrt{x^2-1}}$ for $|x|>1$.

Differentiation Using Substitution

In order to find differential coefficients of complicated expression involving inverse trigonometric functions some substitutions are very helpful, which are listed below.

S.No.	Function	Substitution
1.	$\sqrt{a^2-x^2}$	$x=a\sin\theta$ or $a\cos\theta$
2.	$\sqrt{a^2+x^2}$	$x=a\tan\theta$ or $a\cot\theta$
3.	$\sqrt{x^2-a^2}$	$x=a\sec\theta$ or $a\operatorname{cosec}\theta$
4.	$\sqrt{a+x}$ and $\sqrt{a-x}$	$x=a\cos 2\theta$
5.	$a\sin x+b\cos x$	$a=r\cos\alpha,\ b=r\sin\alpha$
6.	$\sqrt{x-\alpha}$ and $\sqrt{\beta-x}$	$x=\alpha\sin^2\theta+\beta\cos^2\theta$
7.	$\sqrt{2ax-x^2}$	$x=a(1-\cos\theta)$

Differentiation of Implicit Functions

If $f(x, y)=0$ be given implicit function, in which y is not expressible explicitly in term of x, then following steps are very useful to find dy/dx.

Step I Differentiate each term of $f(x, y)=0$

Step II Collect the terms containing dy/dx, and transpose other terms to the right side.

Step III Divide both side of the equation obtained in step II by coefficient of $\frac{dy}{dx}$ to get $\frac{dy}{dx}$.

Differentiation of Parametric Functions

If $x=f(t), y=g(t)$, where t is parameter, then

$$\frac{dy}{dx}=\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}=\frac{\frac{d}{dt}g(t)}{\frac{d}{dt}f(t)}=\frac{g'(t)}{f'(t)}$$

Differentiation of a Function with Respect to Another Function

If $u=f(x)$ and $v=g(x)$, then in order to find derivative of $f(x)$ with respect to $g(x)$. We use following formula $\frac{du}{dv}=\frac{f'(x)}{g'(x)}$.

Logarithmic Differentiation

The process of taking logarithms on both sides and then differentiating them is called logarithmic differentiation. It is useful in the following cases:

(i) If the given function consists of three or more factors which are function of x.

(ii) If the given function is of the form $[f(x)]^{h(x)}$.

(iii) If the given function is of the form $\frac{h_1(x)h_2(x)}{h_3(x)h_4(x)}$.

SPEED BOOSTER

If $y=\{f(x)\}^{h(x)}$,

then $\frac{dy}{dx}=h(x)\{f(x)\}^{h(x)-1}f'(x)+\{f(x)\}^{h(x)}\log f(x)\times h'(x)$

POINTS TO REMEMBER

(i) If $y = f(x)^{\{f(x)\}^{\cdots\infty}}$, then $\frac{dy}{dx} = \frac{y^2 f'(x)}{f(x)\{1 - y\log f(x)\}}$

(ii) If $e^{g(y)} - e^{-g(y)} = 2f(x)$, then $\frac{dy}{dx} = \frac{f'(x)}{g'(y)} \cdot \frac{1}{\sqrt{1 + \{f(x)\}^2}}$

(iii) If $y = \sqrt{\frac{1+g(x)}{1-g(x)}}$, then $\frac{dy}{dx} = \frac{g'(x)}{[1-g(x)]^2} \cdot \sqrt{\frac{1-g(x)}{1+g(x)}}$

(iv) $y = \sqrt{f(x) + \sqrt{f(x) + \sqrt{f(x) + \ldots\infty}}}$, then $\frac{dy}{dx} = \frac{f'(x)}{2y - 1}$

(v) If $\{f(x)\}^{g(y)} = e^{f(x) - g(y)}$, then $\frac{dy}{dx} = \frac{f'(x)\log(x)}{g'(y) \cdot \{1 + \log f(x)\}^2}$

(vi) If $\{f(x)\}^{g(y)} = \{g(y)\}^{f(x)}$, then $\frac{dy}{dx} = \frac{g(y)}{f(x)} \cdot \frac{f'(x)}{g'(y)}\left[\frac{f(x)\log g(y) - g(y)}{g(y)\log f(x) - f(x)}\right]$

Differentiation of Determinant

To differentiate a determinant, we differentiate one row (or column) at a time, keeping others unchanged.

e.g. If $\Delta(x) = \begin{vmatrix} f(x) & g(x) \\ u(x) & v(x) \end{vmatrix}$, then

$$\frac{d}{dx}[\Delta(x)] = \begin{vmatrix} f'(x) & g'(x) \\ u(x) & v(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) \\ u'(x) & v'(x) \end{vmatrix}$$

Note Similar results hold for the differentiation of determinants of higher order.

MASTER STROKES

1. If $x^y = y^x$, $\frac{dy}{dx}$ is equal to

(a) $\frac{y(x\log y - y)}{x(y\log x - x)}$ (b) $\frac{x(\log y - y)}{y(\log x + x)}$ (c) $\frac{y(x\log y + y)}{x(y\log x - x)}$

(d) None of these

2. If $y = \frac{1}{\sqrt{x^2 + a^2} + \sqrt{x^2 + b^2}}$, $\frac{dy}{dx}$ is equal to

(a) $\frac{x}{a^2 - b^2}\left[\frac{1}{\sqrt{x^2 + a^2}} + \frac{1}{\sqrt{x^2 + b^2}}\right]$

(b) $\frac{1}{a^2 - b^2}\left[\frac{1}{\sqrt{x^2 + a^2}} - \frac{1}{\sqrt{x^2 + b^2}}\right]$

(c) $\frac{x}{a^2 - b^2}\left[\frac{1}{\sqrt{x^2 + a^2}} - \frac{1}{\sqrt{x^2 + b^2}}\right]$

(d) $\frac{1}{a^2 - b^2}\left[\frac{1}{\sqrt{x^2 + a^2} + \sqrt{x^2 + b^2}}\right]$

3. If $5f(x) + 3f\left(\frac{1}{x}\right) = x + 2$ and $y = xf(x)$, then $\frac{dy}{dx}$ at $x = 1$ is equal to

(a) 14 (b) 7/8 (c) 1 (d) None of these

4. If $y = a^{x^{x^{x^{\cdots\infty}}}}$, then $x(1 - y\log x\log y)\frac{dy}{dx}$ is

(a) $\frac{y^2}{\log y}$ (b) $\frac{y}{\log y}$ (c) $y^2\log y$ (d) $y\log y$

5. If $f(x) = \sin(\log x)$ and $y = f\left(\frac{2x+3}{3-2x}\right)$, then $\frac{dy}{dx}$ is equal to

(a) $\frac{1}{9 - 4x^2}\cos\left\{\log\left(\frac{2x+3}{3-2x}\right)\right\}$

(b) $\frac{12}{9 - 4x^2}\cos x\left\{\log\left(\frac{2x+3}{3-2x}\right)\right\}$

(c) $\frac{12}{9 - 4x^2}\cos\left\{\log\left(\frac{3-2x}{2x+3}\right)\right\}$

(d) $\frac{12}{9 + 4x^2}\cos\left\{\log\left(\frac{2x+3}{3-2x}\right)\right\}$

6. If $y = (\log_{\cos x}\sin x)(\log_{\sin x}\cos x) + \sin^{-1}\frac{2x}{1+x^2}$, then $\frac{dy}{dx}$ at $x = \pi/2$ is equal to

(a) $\frac{8}{\pi^2 + 4}$ (b) 0 (c) $-\frac{8}{\pi^2 + 4}$ (d) 1

7. If $2^x + 2^y = 2^{x+y}$, then the value of $\frac{dy}{dx}$ at $x = 1$ is

(a) 0 (b) -1 (c) 1 (d) 2

8. If $f(x) = |x - 1|$ and $g(x) = f(f(f(x)))$, then for $x > 2$, $g'(x)$ is equal to

(a) 1 for all $x > 2$ (b) 1 for $2 < x < 3$

(c) -1 for $2 < x < 3$ (d) None of these

9. If $y = (1 + x^{\frac{1}{4}})(1 + x^{\frac{1}{2}})(1 - x^{\frac{1}{4}})$, then $\frac{dy}{dx}$ is

(a) 1 (b) -1 (c) x (d) $\sqrt{x}$

10. If $f(x)$, $g(x)$ and $h(x)$ are three polynomials of degree 2, then $\phi(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ f'(x) & g'(x) & h'(x) \\ f''(x) & g''(x) & h''(x) \end{vmatrix}$

(a) Constant polynomial (b) Polynomial of degree 2

(c) Polynomial of degree (d) None of these

11. If $a_i, b_i \in N$ for $i = 1, 2, 3$ then coefficient of x in the determinant $\begin{vmatrix} (1+x)^{a_1b_1} & (1+x)^{a_1b_2} & (1+x)^{a_1b_3} \\ (1+x)^{a_2b_1} & (1+x)^{a_2b_2} & (1+x)^{a_2b_3} \\ (1+x)^{a_3b_1} & (1+x)^{a_3b_2} & (1+x)^{a_3b_3} \end{vmatrix}$ is

(a) 0 (b) 1 (c) 2 (d) 3

Answers

1. (a) 2. (c) 3. (b) 4. (c) 5. (b)
6. (c) 7. (b) 8. (c) 9. (b) 10. (a)
11. (a)

Learn Through Problems

SEQUENCES AND SERIES

Section targetted to provide stepwise learning & practice through selective problems

1 The sum of the series $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{n(n+1)}$ is

(a) $\frac{1}{n+1}$ (b) $1-\frac{1}{n+1}$ (c) $\frac{1}{n+1}-1$ (d) $1+\frac{1}{n+1}$

(b) ***Step 1*** Write the general term (T_r) of the given sequence

$$T_r=\frac{1}{r(r+1)};\ r=1, 2, \ldots, n$$

Step 2 Simplify the general term expression

$$T_r=\frac{1}{r}-\frac{1}{r+1};\ r=1, 2, \ldots, n$$

Step 3 Use the formula of S_n, i.e.

$S_n=\sum_{r=1}^{n}T_r$ and solve it.

$$\therefore\quad S_n=\sum_{r=1}^{n}T_r=\sum_{r=1}^{n}\left(\frac{1}{r}-\frac{1}{r+1}\right)$$

$$=\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\ldots+\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

$$=1-\frac{1}{n+1}$$

TRY THESE

1. The sum to n terms of the series $\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\ldots$ is

(a) $\frac{1}{2n+1}$ (b) $\frac{2n}{2n+1}$ (c) $\frac{n}{2n+1}$ (d) $\frac{2n}{n+1}$

2. If $t_n=\frac{1}{4}(n+2)(n+3)$ for $n=1, 2, 3, \ldots$, then $\frac{1}{t_1}+\frac{1}{t_2}+\frac{1}{t_3}+\ldots+\frac{1}{2003}$ is equal to

(a) $\frac{4006}{3006}$ (b) $\frac{4003}{3007}$ (c) $\frac{4006}{3008}$ (d) $\frac{4006}{3009}$

3. The limiting value of the sum to n terms of the series $\frac{3}{1^2\cdot 2^2}+\frac{5}{2^2\cdot 3^2}+\frac{7}{3^2\cdot 4^2}+\ldots$ as $n\to\infty$ is

(a) 0 (b) 2 (c) $\frac{1}{2}$ (d) 1

1. (c) **2.** (d) **3.** (d)

2 Find the sum to n terms of the series

$$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\ldots+\frac{1}{(2n-1)(2n+1)}$$

Step 1 Check the difference between the two numbers which are in multiplication in denominator.

$$3-1, 5-3, 5-7, \ldots, (2n+1)-(2n-1)$$
$$=2, 2, 2, \ldots, 2$$

Step 2 If difference is same make the difference in numerator by multiplying and dividing whole series by that difference.

$$\frac{1}{2}\left[\frac{2}{1\cdot 3}+\frac{2}{3\cdot 5}+\frac{2}{5\cdot 7}+\ldots+\frac{2}{(2n-1)(2n+1)}\right]$$

Step 3 Write each term as the difference of two terms as given below $\frac{1}{a(a+d)}=\frac{1}{d}\left(\frac{1}{a}-\frac{1}{a+d}\right)$

$$\frac{1}{2}\left[\left(\frac{1}{1}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{7}\right)+\ldots+\left(\frac{1}{2n-1}-\frac{1}{2n+1}\right)\right]$$

Step 4 Simplify the equation and get the required sum

$$=\frac{1}{2}\left[\frac{1}{1}-\frac{1}{3}+\frac{1}{3}-\frac{1}{5}+\frac{1}{5}-\frac{1}{7}+\ldots+\frac{1}{2n-1}-\frac{1}{2n+1}\right]$$

$$=\frac{1}{2}\left[1-\frac{1}{2n+1}\right]=\frac{1}{2}\left[\frac{2n+1-1}{2n+1}\right]=\frac{n}{2n+1}$$

TRY THESE

1. The sum to n terms of the series $\frac{3}{1^2\cdot 2^2}+\frac{5}{2^2\cdot 3^2}+\frac{7}{3^2\cdot 4^2}+\ldots$ is

(a) $\frac{2n+n^2}{(n+1)^2}$ (b) $\frac{2n}{(n+1)^2}$

(c) $\frac{2n-n^2}{(n+1)^2}$ (d) None of these

2. $\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\frac{1}{7\cdot 10}+\ldots$ upto n terms is

(a) $\frac{n}{n+1}$ (b) $\frac{n}{3n+1}$ (c) $\frac{n+1}{n+4}$ (d) $\frac{2n}{3n+1}$

3. $\frac{1}{1\cdot 6}+\frac{1}{6\cdot 11}+\frac{1}{11\cdot 16}+\frac{1}{16\cdot 21}+\ldots+\frac{1}{(5n-4)(5n+1)}$ is equal to

(a) $\frac{5n+1}{n+2}$ (b) $\frac{n+2}{5n+1}$ (c) $\frac{n}{5n+1}$ (d) $\frac{n-1}{5n+1}$

1. (a) **2.** (b) **3.** (c)

3 Find the sum to n terms of the series

$$1 + 5 + 12 + 22 + 35 + \ldots$$

Step 1 Obtain the terms of the series and compute the difference between $T_2 - T_1, T_3 - T_2, T_4 - T_3, \ldots$ and so on.

$$5 - 1, 12 - 5, 22 - 12, 35 - 22, \ldots = 4, 7, 10, 13, \ldots$$

Step 2 If the differences are in AP, then take the nth term as $T_n = an^2 + bn + c$. If the difference are in GP, then take the nth term or $T_n = ar^{n-1} + bn + c$, where r is common ratio. If the difference of difference is in AP, then take $T_n = an^3 + bn^2 + cn + d$. If the difference of difference are in GP, then take $T_n = ar^{n-1} + bn^2 + cn + d$

Here, difference are in AP.

$\therefore$ $T_n = an^2 + bn + c$

Step 3 Put $n = 1, 2, 3$ in T_n and calculate the values of a, b, c.

Putting $n = 1, 2, 3$, we get

$T_1 = a + b + c \Rightarrow a + b + c = 1$ $[\because T_1 = 1]$

$T_2 = 4a + 2b + c \Rightarrow 4a + 2b + c = 5$ $[\because T_2 = 5]$

$T_3 = 9a + 3b + c \Rightarrow 9a + 3b + c = 12$ $[\because T_3 = 12]$

On solving these equations, we get

$$a = \frac{3}{2}, b = -\frac{1}{2} \text{ and } c = 0$$

Step 4 Substitute the values of a, b, c in T_n and get an expression for T_n

$\therefore$ $T_n = \frac{3}{2}n^2 - \frac{1}{2}n = \frac{1}{2}(3n^2 - n)$

Step 5 Use the formula for S_n, i.e. $S_n = \sum_{r=1}^{n} T_r$

$$\therefore \quad S_n = \sum_{r=1}^{n} T_r = \sum_{r=1}^{n} \frac{1}{2}(3r^2 - r) = \frac{1}{2}\left(\sum_{r=1}^{n} 3r^2 - \sum_{r=1}^{n} r\right)$$

$$= \frac{3}{2}\sum_{r=1}^{n} r^2 - \frac{1}{2}\sum_{r=1}^{n} r$$

$$= \frac{3}{2}\left[\frac{n(n+1)(2n+1)}{6}\right] - \frac{1}{2}\left[\frac{n(n+1)}{2}\right] = \frac{n^2(n+1)}{2}$$

TRY THESE

1. The sum of series upto n terms is equal to $3 + 7 + 13 + 21 + 31 + \ldots$

(a) $\frac{n(n^2 + 3n + 5)}{3}$ (b) $\frac{n(n+1)(n+2)}{6}$

(c) $\frac{n^2(n+1)}{9}$ (d) $\frac{n(n^2 + 6n + 2)}{3}$

2. $5 + 7 + 13 + 31 + 85 + \ldots$ upto n terms is equal to

(a) $\frac{3^n - 1}{2}$ (b) $\frac{3^n + 6n + 2}{2}$

(c) $\frac{3^n + 8n - 1}{2}$ (d) $\frac{3^n + 6n + 2}{6}$

3. $5 + 11 + 19 + 29 + 41 + \ldots$ upto n terms is equal to

(a) $\frac{n(n+1)(n+4)}{2}$ (b) $\frac{n(n+2)(n+6)}{6}$

(c) $\frac{n(n+1)(n+2)}{9}$ (d) $\frac{n(n+2)(n+4)}{3}$

1. (a) **2.** (c) **3.** (d)

4 Find the sum of the given series upto n terms

$$5 + 55 + 555 + \ldots .$$

Step 1 Take common the number in the given series.

$$5[1 + 11 + 111 + \ldots \text{ to } n \text{ terms}]$$

Step 2 Multiply and divide by 9

$$\frac{5}{9}[9 + 99 + 999 + \ldots \text{ to } n \text{ terms}]$$

Step 3 Replace 9 by $(10 - 1)$, 99 by $(10^2 - 1)$ and so on.

$$\frac{5}{9}[(10 - 1) + (10^2 - 1) + \ldots \text{ to } n \text{ terms}]$$

Step 4 Make two sequence one $10, 10^2, 10^3, \ldots$ and another $-1, -1, -1, \ldots$.

$$\frac{5}{9}[(10 + 10^2 + 10^3 + \ldots + 10^n) - (1 + 1 + 1 + \ldots n \text{ terms})]$$

Step 5 Apply the sum of n terms of GP and simplify it.

$$\frac{5}{9}\left[\frac{10(10^n - 1)}{10 - 1} - n\right] = \frac{5}{9}\left[\frac{10}{9}(10^n - 1) - n\right]$$

$$= \frac{5}{81}[10^{n+1} - 10 - 9n]$$

TRY THESE

1. The sum of the series $0.6 + 0.66 + 0.666 + \ldots$ to n terms is equal to

(a) $\frac{2}{3}n + \frac{2}{27}(1 - 10^{-n})$ (b) $\frac{2}{3}n - \frac{2}{27}(1 - 10^{-n})$

(c) $\frac{2}{3}n - \frac{2}{27}(1 - 10^{n})$ (d) $\frac{2}{3}n + \frac{2}{27}(1 - 10^{n})$

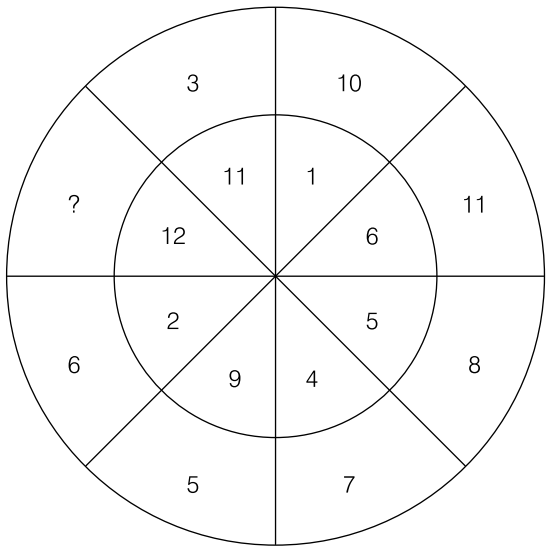
2. $7 + 77 + 777 + \ldots$ to n terms is equal to

(a) $\frac{7}{81}[10^{n+1} - 10 - 9n]$ (b) $\frac{7}{81}[10^{n+1} + 10 - 9n]$

(c) $\frac{7}{9}[10^{n+1} - 10 - 9n]$ (d) $\frac{7}{81}[10^{n+1} - 10 - 7n]$

1. (b) **2.** (a)

CAN YOU GUESS THE NUMBER?



TARGET JEE 2017

JEE ADVANCED DRILL

A collection of 30 innovative problems on JEE Advanced pattern from complete syllabus

Single Option Correct Type

1. If the lines $\frac{x-4}{1}=\frac{y-2}{1}=\frac{z-\lambda}{3}$ and $\frac{x}{1}=\frac{y+2}{2}=\frac{z}{4}$ intersect each other, then λ lies in the interval

(a) (9, 11) (b) (–5, –3)
(c) (13, 15) (d) (11, 13)

2. Consider a branch of the hyperbola $x^2-2y^2-2\sqrt{2}x, -4\sqrt{2}y-6=0$ with vertex at the point A. Let B be one of the end-points of its latusrectum. If C is the focus of the hyperbola nearest to the point A, then the area of ΔABC is

(a) $1-\sqrt{\frac{2}{3}}$ (b) $\sqrt{\frac{3}{2}}+1$
(c) $1+\sqrt{\frac{2}{3}}$ (d) $\sqrt{\frac{3}{2}}-1$

3. If $2^{2\pi/\sin^{-1}x}-2(a+2)2^{\pi/\sin^{-1}x}+8a<0$ for atleast one real x, then

(a) $\frac{1}{8}\le a<2$
(b) $a<2$
(c) $a\in R-\{2\}$
(d) $a\in\left[0,\frac{1}{8}\right)\cup(2,\infty)$

4. A person buys eight packets of TIDE detergent. Each packet contains one coupon, which bears one of the letters of the word TIDE. If he shows all the letters of the word TIDE, he gets one free packet. If he gets exactly one free packet, then the number of different possible combinations of the coupons is

(a) ${}^7C_3$ (b) ${}^8C_4-1$
(c) ${}^8C_3$ (d) 4^4

5. If p and q are distinct prime numbers, then the number of distinct imaginary numbers which are pth as well as qth roots of unity are

(a) min (p, q) (b) max (p, q)
(c) zero (d) 1

One or More than One Option Correct Type

6. If $3^x=4^{x-1}$, then x is equal to

(a) $\frac{2\log_3 2}{2\log_3 2-1}$ (b) $\frac{2}{2-\log_2 3}$
(c) $\frac{1}{1-\log_4 3}$ (d) $\frac{2\log_2 3}{2\log_2 3-1}$

7. If $0\le x\le 2\pi$ and $|\cos x|\le\sin x$, then

(a) the set of all values of x is $\left[\frac{\pi}{4},\frac{3\pi}{4}\right]$
(b) the number of solutions that are integral multiple of $\frac{\pi}{4}$ is four
(c) the sum of the largest and the smallest solution is π
(d) the set of all values of x is $x\in\left[\frac{\pi}{4},\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\frac{3\pi}{4}\right]$

8. The 10th term of $\left(3-\sqrt{\frac{17}{4}+3\sqrt{2}}\right)^{20}$ is

(a) an irrational number
(b) a rational number
(c) a positive integer
(d) a negative integer

9. $\int_0^x\{\int_0^u f(t)dt\}du$ is equal to

(a) $\int_0^x(x-u)f(u)du$ (b) $\int_0^x uf(x-u)du$
(c) $x\int_0^x f(u)du$ (d) $x\int_0^x uf(u-x)du$

10. If $y = ae^{-1/x} + b$ is a solution of $\frac{dy}{dx} = \frac{y}{x^2}$, then

(a) $a \in R$
(b) $b = 0$
(c) $b = 1$
(d) a takes finite number of values

11. If the normal at P to the rectangular hyperbola $x^2 - y^2 = 4$ meets the axes at G and g, C is the centre of the hyperbola, then

(a) $PG = PC$ (b) $Pg = PC$ (c) $PG = Pg$ (d) $Gg = 2PC$

12. A vector $\mathbf{d}$ is equally inclined to three vectors $\mathbf{a} = \hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$, $\mathbf{b} = 2\hat{\mathbf{i}} + \hat{\mathbf{j}}$ and $\mathbf{c} = 3\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$. Let $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ be three vectors in the plane of $\mathbf{ab}$; $\mathbf{bc}$; $\mathbf{ca}$, respectively. Then,

(a) $\mathbf{z} \cdot \mathbf{d} = 0$
(b) $\mathbf{x} \cdot \mathbf{d} = 1$
(c) $\mathbf{y} \cdot \mathbf{d} = 32$
(d) $\mathbf{r} \cdot \mathbf{d} = 0$, where $\mathbf{r} = \lambda\mathbf{x} + \mu\mathbf{y} + \gamma\mathbf{z}$

13. Which of the following values of α satisfy the equation $\begin{vmatrix} (1+\alpha)^2 & (1+2\alpha)^2 & (1+3\alpha)^2 \\ (2+\alpha)^2 & (2+2\alpha)^2 & (2+3\alpha)^2 \\ (3+\alpha)^2 & (3+2\alpha)^2 & (3+3\alpha)^2 \end{vmatrix} = -648\alpha$?

(a) –4 (b) 9 (c) –9 (d) 4

Single Integer Type

14. Let $A = [a_{ij}]_{3\times 3}$ be a matrix such that $AA^T = 4I$ and $a_{ij} + 2c_{ij} = 0$, where c_{ij} is the cofactor of a_{ij} and I is the unit matrix of order 3.

$$\begin{vmatrix} a_{11}+4 & a_{12} & a_{13} \\ a_{21} & a_{22}+4 & a_{23} \\ a_{31} & a_{32} & a_{33}+4 \end{vmatrix} + 5\lambda \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22}+1 & a_{23} \\ a_{31} & a_{32} & a_{33}+1 \end{vmatrix} = 0,$$

then the value of 10 λ is .....

15. Let $g(x) = \begin{cases} a\sqrt{x+1}, & \text{if } 0 < x < 3 \\ bx + 2, & \text{if } 3 \le x < 5 \end{cases}$, if $g(x)$ is differentiable on $(0, 5)$, then $(a + b)$ is equal to .....

16. If $y = \frac{a + bx^{3/2}}{x^{5/4}}$ and $y' = 0$ at $x = 5$, then the value of $\frac{a^2}{b^2}$ is ..... .

17. The value of $6 + \log_{3/2}\left(\frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}\ldots}}}\right)$ is ..... .

18. If $\cos 4x = a_0 + a_1 \cos^2 x + a_2 \cos^4 x$ is true for all values of $x \in R$, then the value of $5a_0 + a_1 + a_2$ is ...

19. Two parallel chords of a circle of radius 2 are at a distance $\sqrt{3} + 1$ a part. If the chord subtend angles $\frac{\pi}{k}$ and $\frac{\pi}{2k}$ at the centre, where $k > 0$, then the value of $[k]$ is .....($[k]$ denotes the largest integer less than or equal to k).

20. The line $x = c$ cuts the triangle with vertices (0, 0), (1, 1) and (9, 1) into two regions.

For the areas of the two regions to be the same, c must be equal to..... .

21. A vertical line passing through the point $(h, 0)$ intersects the ellipse $\frac{x^2}{4} + \frac{y^2}{3} = 1$ at points P and Q. Let the tangents to the ellipse at P and Q meet at point R. If $\Delta(h)$ = area of ΔPQR, $\Delta_1 = \max_{1/2 \le h \le 1} \Delta h$, and $\Delta_2 = \min_{1/2 \le h \le 1} \Delta h$, then $\frac{8}{\sqrt{5}}\Delta_1 - 8\Delta_2$ is equal to...

Linked Comprehension Type

Directions (Q. Nos. 22-23) A ray of light comes along the line $L = 0$ and strikes the plane mirror kept long the plane $P = 0$ at B. $A(2, 1, 6)$ is a point on the line $L = 0$, whose image about $P = 0$ is A'. It is given that $L = 0$ is $\frac{x-2}{3} = \frac{y-1}{4} = \frac{z-6}{5}$ and $P = 0$ is $x + y - 2z = 3$.

22. The coordinates of A' are

(a) (6, 5, 2) (b) (6, 5, –2)
(c) (6, – 5, 2) (d) None of these

23. The coordinates of B are

(a) (5, 10, 6) (b) (10, 15, 11)
(c) (–10, – 15, – 14) (d) None of these

Directions (Q. Nos. 24-25)
Given, $|px^2 + qx + r| \le |Px^2 + Qx + R|, \forall x \in R$ and $d = q^2 - 4pr > 0$ and $D = Q^2 - 4PR > 0$

24. Which of the following must be true?

(a) $|p| \ge |P|$ (b) $|p| \le |P|$
(c) $|p| = |P|$ (d) All of these

25. Which of the following must be true?

(a) $|d| \le |D|$ (b) $|d| \ge |D|$
(c) $|d| = |D|$ (d) None of these

Directions (Q. Nos. 26-27) The line $x + 2y + a = 0$ intersects the circle $x^2 + y^2 = 4$ at two distincts points A and B. Another line $12x - 6y - 41 = 0$, intersects the circle $x^2 + y^2 - 4x - 2y + 1 = 0$ at two distinct points C and D.

26. The value of a for which the four points A, B, C and D are concyclic is

(a) 1 (b) 3
(c) 4 (d) 2

27. The equation of the circle passing through the points A, B, C and D is

(a) $5x^2 + 5y^2 - 8x - 16y - 36 = 0$

(b) $5x^2 + 5y^2 + 8x - 16y - 36 = 0$

(c) $5x^2 + 5y^2 + 8x + 16y - 36 = 0$

(d) $5x^2 + 5y^2 - 8x - 16y + 36 = 0$

Match the Columns

28. Match the following columns.

	Column I	Column II
A.	If $I = \int_{-2}^{2}(\alpha x^3 + \beta x + \gamma)dx$, then I is	p. independent of α
B.	Let α, β be the distinct positive roots of the equation $\tan x = 2x$. Then, $\gamma\int_0^1 (\sin\alpha x \cdot \sin\beta x)dx$ (where $\gamma \neq 0$) is	q. independent of β
C.	If $(x + \alpha) + f(x) = 0$, where $\alpha > 0$, then $\int_{\beta}^{\beta + 2\gamma\alpha} f(x)dx$, where $\gamma \in N$, is	r. independent of γ
D.	$\gamma\int_0^{\alpha}[\sin x]dx$ is, where $\gamma \neq 0$, $\alpha \in [(2\beta + 1)\pi, (2\beta + 2)\pi], \beta \in N$, and where $[.]$ denotes the greatest integer function.	s. depends on α

29. Match the following columns.

	Column I	Column II
A.	If a, b, c are in GP, then $\log_a 10$, $\log_b 10$, $\log_c 10$ are in	p. AP
B.	If $\frac{a + be^x}{a - be^x} = \frac{b + ce^x}{b - ce^x} = \frac{c + de^x}{c - de^x}$, then a, b, c, d are in	q. HP
C.	If a, b, c are in AP; a, x, b are in GP and b, y, c are in GP, then x^2, b^2, y^2 are in	r. GP
D.	If x, y, z are in GP and $a^x = b^y = c^z$, then $\log a$, $\log b$, $\log c$ are in	s. None of these

30. Let $f(x) = \frac{x^2 - 6x + 5}{x^2 - 5x + 6}$

Match the expressions/statements in Column I with expression/statements in Column II.

	Column I	Column II
A.	If $-1 < x < 1$, then $f(x)$ satisfies	p. $0 < f(x) < 1$
B.	If $1 < x < 2$, then $f(x)$ satisfies	q. $f(x) < 0$
C.	If $3 < x < 5$, then $f(x)$ satisfies	r. $f(x) > 0$
D.	If $x > 5$, then $f(x)$ satisfies	s. $f(x) < 1$

Answers with Explanation

1. ***(d)*** Given, lines will intersect, if

$$\begin{vmatrix} 4-0 & 2-(-2) & \lambda - 0 \\ 1 & 1 & 3 \\ 1 & 2 & 4 \end{vmatrix} = 0$$

$\Rightarrow 4(4-6) - 4(4-3) + \lambda(2-1) = 0$

$\Rightarrow \lambda = 12 \Rightarrow \lambda \in (11, 13)$

2. ***(d)*** The given hyperbola is $\frac{(x - \sqrt{2})^2}{4} - \frac{(y + \sqrt{2})^2}{2} = 1$

Here, $a = 2, b = \sqrt{2}, e = \sqrt{\frac{3}{2}}$

Therefore, the required area $= \frac{1}{2}a(e-1) \times \frac{b^2}{a}$

$$= \frac{1}{2} \cdot \frac{(\sqrt{3} - \sqrt{2}) \times 2}{\sqrt{2}} \times \frac{(\sqrt{2})^2}{2}$$

$$= \frac{\sqrt{3} - \sqrt{2}}{\sqrt{2}} = \sqrt{\frac{3}{2}} - 1$$

3. ***(d)*** $2^{2\pi/\sin^{-1}x} - 2(a+2)2^{\pi/\sin^{-1}x} + 8a < 0$

$(2^{\pi/\sin^{-1}x} - 4)(2^{\pi/\sin^{-1}x} - 2a) < 0$

Now, $2^{\pi/\sin^{-1}x} \in \left(0, \frac{1}{4}\right] \cup [4, \infty)$

For $2^{\pi/\sin^{-1}x} \in \left(0, \frac{1}{4}\right]$, we have

$(2^{\pi/\sin^{-1}x} - 4) < 0$

$\therefore$ $2^{\pi/\sin^{-1}x} - 2a > 0$ or $2a < 2^{\pi/\sin^{-1}x}$

or $2a < \frac{1}{4}$ or $0 \le a < \frac{1}{8}$

Similarly, for $2^{\pi/\sin^{-1}x} \in [4, \infty)$, $a > 2$, we get

$$a \in \left[0, \frac{1}{8}\right) \cup (2, \infty)$$

4. ***(a)*** Let x_1, x_2, x_3, x_4 be the number of times T, I, D, E appears on the coupon. Then, we must have $x_1 + x_2 + x_3 + x_4 = 8$, where $1 \le x_1, x_2, x_3, x_4 \le 8$ (as each letter must appear once). Then, the required number of combinations of coupons is equivalent to the number of positive integral solutions of the above equation, which is further equivalent to number of ways of 8 identical objects distributed among 4 persons when each gets atleast one object and is given by ${}^{8-1}C_{4-1} = {}^{7}C_3$.

5. ***(c)*** We have,

$$e^{2\pi ri/p} = e^{2\pi mi/q}$$

$r = 0, 1, \ldots, p-1$

$m = 0, 1, \ldots, q-1$

This is possible iff $r = m = 0$ but for $r = m = 0$ we get 1, which is not an imaginary number.

6. ***(a, b, c)*** Given, $3^x = 4^{x-1}$

$\Rightarrow \log_2 3^x = (x-1)\log_2 4 = 2(x-1)$

$\Rightarrow x\log_2 3 = 2x - 2$

$\Rightarrow x = \frac{2}{2 - \log_2 3}$

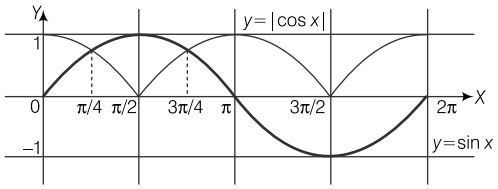
Rearranging, we get

$$x = \frac{2}{2 - \frac{1}{\log_3 2}} = \frac{2\log_3 2}{2\log_3 2 - 1}$$

Rearranging again, we get

$$x = \frac{\log_3 4}{\log_3 4 - 1} = \frac{\frac{1}{\log_4 3}}{\frac{1}{\log_4 3} - 1} = \frac{1}{1 - \log_4 3}$$

7. ***(a, c)*** It is easier to solve the inequality using graphical method. The graphs of $y = |\cos x|$ and $y = \sin x$ are as shown in the following figure.



From the figure, $|\cos x| \le \sin x$ for $x \in \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$

8. ***(a)*** We have,

$$\frac{17}{4} + 3\sqrt{2} = \frac{1}{4}(9 + 8 + 12\sqrt{2}) = \frac{1}{4}(3 + 2\sqrt{2})^2$$

$$\therefore \quad 3 - \sqrt{\frac{17}{4} + 3\sqrt{2}} = 3 - \frac{1}{2}(3 + 2\sqrt{2}) = \frac{3}{2} - \sqrt{2}$$

Hence, the 10th term of

$$\left(3 - \sqrt{\frac{17}{4} + 3\sqrt{2}}\right)^{20} = \left(\frac{3}{2} - \sqrt{2}\right)^{20} \text{ is } {}^{20}C_9\left(\frac{3}{2}\right)^{20-9}(-\sqrt{2})^9$$

which is an irrational number.

9. ***(a, b)*** Let $I = \int_0^x \{\int_0^u f(t)dt\}du$

Integrating by parts, choose 1 as the second function. Then,

$$I = \{u\int_0^u f(t)dt\}_0^x - \int_0^x f(u)u\,du$$

$$= x\int_0^x f(t)dt - \int_0^x f(u)\,u\,du$$

$$= x\int_0^x f(u)du - \int_0^x f(u)\,u\,du = \int_0^x f(u)(x-u)\,du$$

Again, let $u = x - z$

$\Rightarrow \quad du = -dz$

$\therefore I = -\int_x^0 f(x-z)z\,dz = \int_0^x zf(x-z)dz = \int_0^x uf(x-u)\,du$

10. ***(a, b)*** $\frac{dy}{dx} = \frac{y}{x^2}$ or $\frac{dy}{y} = \frac{dx}{x^2}$

or $\log y = \frac{-1}{x} + \log C$

$\Rightarrow \quad \frac{y}{C} = e^{-1/x} \Rightarrow y = Ce^{-1/x}$

Comparing with $y = ae^{-1/x} + b, a \in R, b = 0$

11. ***(a, b, c, d)*** Normal at point $P(2\sec\theta, 2\tan\theta)$ is

$$\frac{2x}{\sec\theta} + \frac{2y}{\tan\theta} = 8$$

It meets the axes at points $G(4\sec\theta, 0)$ and $g(0, 4\tan\theta)$.

Then,

$$PG = \sqrt{4\sec^2\theta + 4\tan^2\theta}$$

$$Pg = \sqrt{4\sec^2\theta + 4\tan^2\theta}$$

$$PC = \sqrt{4\sec^2\theta + 4\tan^2\theta}$$

$$Gg = \sqrt{16\sec^2\theta + 16\tan^2\theta}$$

$$= 2\sqrt{4\sec^2\theta + 4\tan^2\theta} = 2PC$$

12. ***(a, d)*** $\mathbf{a} = \hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$,

$\mathbf{b} = 2\hat{\mathbf{i}} + \hat{\mathbf{j}}$ and $\mathbf{c} = 3\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$

Since, $[\mathbf{a}\,\mathbf{b}\,\mathbf{c}] = \begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & 0 \\ 0 & 3 & -2 \end{vmatrix} = 0$

Therefore, $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are coplanar vectors. Further since $\mathbf{d}$ is equally inclined to $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, we have

$\mathbf{d}\cdot\mathbf{a} = \mathbf{d}\cdot\mathbf{b} = \mathbf{d}\cdot\mathbf{c} = 0$

$\therefore \quad \mathbf{d}\cdot\mathbf{x} = \mathbf{d}\cdot\mathbf{y} = \mathbf{d}\cdot\mathbf{z} = 0$

$\therefore \quad \mathbf{d}\cdot\mathbf{r} = 0$

13. ***(b, c)*** We have,

$$\begin{vmatrix} (1+\alpha)^2 & (1+2\alpha)^2 & (1+3\alpha)^2 \\ (2+\alpha)^2 & (2+2\alpha)^2 & (2+3\alpha)^2 \\ (3+\alpha)^2 & (3+2\alpha)^2 & (3+3\alpha)^2 \end{vmatrix} = -648\alpha$$

Applying $R_2 \to R_2 - R_1, R_3 \to R_3 - R_2$

$$\begin{vmatrix} (1+\alpha)^2 & (1+2\alpha)^2 & (1+3\alpha)^2 \\ (3+2\alpha) & (3+4\alpha) & (3+6\alpha) \\ (5+2\alpha) & (5+4\alpha) & (5+6\alpha) \end{vmatrix} = -648\alpha$$

Applying $R_3 \to R_3 - R_2$

$$\begin{vmatrix} (1+\alpha)^2 & (1+2\alpha)^2 & (1+3\alpha)^2 \\ (3+2\alpha) & (3+4\alpha) & (3+6\alpha) \\ 2 & 2 & 2 \end{vmatrix} = -648\alpha$$

Applying $C_2 \to C_2 - C_1, C_3 \to C_3 - C_2$

$$\begin{vmatrix} (1+\alpha)^2 & \alpha(2+3\alpha) & \alpha(2+5\alpha) \\ (3+2\alpha) & 2\alpha & 2\alpha \\ 2 & 0 & 0 \end{vmatrix} = -648\alpha$$

$\Rightarrow \quad 2\alpha^2(2 + 3\alpha) - 2\alpha^2(2 + 5\alpha) = -324\alpha$

$\Rightarrow \quad -4\alpha^3 = -324\alpha$

$\Rightarrow \quad \alpha = 0, \pm 9$

14. ***(4)*** Given that, $AA^T = 4I$

$\Rightarrow \quad |A|^2 = 4$

or $\quad |A| = \pm 2$

So, $\quad A^T = 4\,A^{-1} = 4\frac{\text{adj}\,(A)}{|A|}$

$$\Rightarrow \begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{vmatrix} = \frac{4}{|A|}\begin{vmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{vmatrix}$$

Now, $\quad a_{ij} = \frac{4}{|A|}c_{ij}$

$\Rightarrow \quad -2c_{ij} = \dfrac{4}{|A|}c_{ij} \qquad [\text{as } a_{ij} + 2c_{ij} = 0]$

$\Rightarrow \quad |A| = -2$

Now, $|A + 4I| = |A + AA^T| = |A||I + A^T|$

$= -2|(I + A)^T|$

$= -2|I + A|$

$\Rightarrow |A + 4I| + 2|A + I| = 0$

So, on comparing, we get $5\lambda = 2$

$\Rightarrow \quad \lambda = \dfrac{2}{5}$

Hence, $10\lambda = 4$

15. ***(2)*** $g'(3^-) = \lim\limits_{h\to 0}\dfrac{g(3-h)-g(3)}{-h}$

$= \lim\limits_{h\to 0}\dfrac{a\sqrt{4-h}-(3b+2)}{-h} \qquad \ldots(i)$

For existence of limit, $\lim\limits_{h\to 0} = 0$

$\therefore \quad 2a - 3b = 2 \qquad \ldots(ii)$

Now,

$g'(3^+) = \lim\limits_{h\to 0}\dfrac{b(3+h)+2-(3b+2)}{h} = b \qquad \ldots(iii)$

Substituting $3b + 2 = 2a$ in Eq. (i), we get

$g'(3^-) = \lim\limits_{h\to 0}\dfrac{a\sqrt{4-h}-2a}{-h}$

$= \lim\limits_{h\to 0}\left(\dfrac{(4-h)-4}{(-h)\sqrt{4-h}+2}\right) = \dfrac{a}{4}$

Hence, $g'(3^-) = g'(3^+)$

$\dfrac{a}{4} = b \Rightarrow a = 4b \qquad \ldots(iv)$

From Eqs. (ii) and (iv), we get

$8b - 3b = 2$

$\Rightarrow \quad b = \dfrac{2}{5}$ and $a = \dfrac{8}{5}$

$\therefore \quad a + b = 2$

16. ***(5)*** $y = \dfrac{a + bx^{3/2}}{x^{5/4}}$

or $y' = \dfrac{\frac{3}{2}bx^{1/2}x^{5/4} - \frac{5}{4}x^{1/4}(a + bx^{3/2})}{x^{5/2}}$

According to the question,

$0 = \dfrac{\frac{3}{2}b\,5^{1/2}\,5^{5/4} - \frac{5}{4}5^{1/4}(a + b\cdot 5^{3/2})}{5^{5/2}}$

$\Rightarrow \quad \dfrac{3b}{2}5^{7/4} - a\dfrac{5^{5/4}}{4} - 5b\dfrac{5^{7/4}}{4} = 0$

$\Rightarrow \quad b5^{7/4} = a5^{5/4} \Rightarrow b\sqrt{5} = a$

$\Rightarrow \quad a : b = \sqrt{5} : 1$

$\therefore \quad \dfrac{a^2}{b^2} = \dfrac{5}{1} = 5$

17. ***(4)*** Let $\sqrt{4 - \dfrac{1}{3\sqrt{2}}\sqrt{4 - \dfrac{1}{3\sqrt{2}}\sqrt{4 - \dfrac{1}{3\sqrt{2}}\ldots}}} = y$

So, $4 - \dfrac{1}{3\sqrt{2}}y = y^2 \qquad [\because y > 0]$

$\Rightarrow \quad y^2 + \dfrac{1}{3\sqrt{2}}y - 4 = 0 \Rightarrow y = \dfrac{8}{3\sqrt{2}}$

So, the required value

$= 6 + \log_{3/2}\left(\dfrac{1}{3\sqrt{2}} \times \dfrac{8}{3\sqrt{2}}\right) = 6 + \log_{3/2}\dfrac{4}{9} = 6 - 2 = 4$

18. ***(5)*** $\cos 4x = 2\cos^2 2x - 1 = 2(2\cos^2 x - 1)^2 - 1$

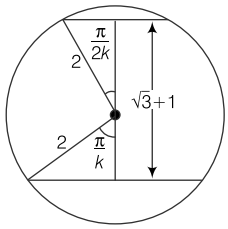
$= 2(4\cos^4 x + 1 - 4\cos^2 x) - 1$

$= 8\cos^4 x - 8\cos^2 x + 1$

$\therefore \quad a_0 = 1, a_1 = -8, a_2 = 8$

$\therefore \quad 5a_0 + a_1 + a_2 = 5$

19. ***(3)***



$2\cos\dfrac{\pi}{2k} + 2\cos\dfrac{\pi}{k} = \sqrt{3} + 1$

$\Rightarrow \quad \cos\dfrac{\pi}{2k} + \cos\dfrac{\pi}{k} = \dfrac{\sqrt{3}+1}{2}$

Let $\dfrac{\pi}{k} = \theta$. Then, $\cos\theta + \cos\dfrac{\theta}{2} = \dfrac{\sqrt{3}+1}{2}$

$\Rightarrow 2\cos^2\dfrac{\theta}{2} - 1 + \cos\dfrac{\theta}{2} = \dfrac{\sqrt{3}+1}{2}$

$\Rightarrow \quad 2t^2 + t - \dfrac{\sqrt{3}+3}{2} = 0 \qquad \left[\text{where, } \cos\left(\dfrac{\theta}{2}\right) = t\right]$

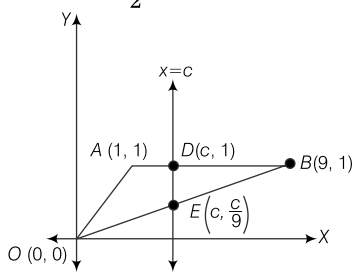
$\Rightarrow \quad t = \dfrac{-1 \pm \sqrt{1 + 4(3+\sqrt{3})}}{4}$

$= \dfrac{-1 \pm (2\sqrt{3}+1)}{4} = \dfrac{-2-2\sqrt{3}}{4}, \dfrac{\sqrt{3}}{2}$

$\Rightarrow \quad t = \cos\dfrac{\theta}{2} \in [-1, 1]$

$\therefore \quad \cos\dfrac{\theta}{2} = \dfrac{\sqrt{3}}{2} \Rightarrow \dfrac{\theta}{2} = \dfrac{\pi}{6} \Rightarrow k = 3$

20. ***(3)*** Area of $\Delta OAB = \dfrac{1}{2}(1)(8) = 4$ sq units



The equation of OB is $y = \dfrac{1}{9}x$

Hence, the point E is $(c, c/9)$.

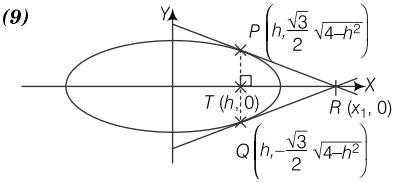
Now, the area of ΔBDE is 2 sq units. Therefore,

$\dfrac{1}{2}\left(1 - \dfrac{c}{9}\right)(9 - c) = 2$

$\Rightarrow \quad (9 - c)^2 = 36$

$\Rightarrow \quad 9 - c = \pm 6 \Rightarrow c = 3$

21. ***(9)***



$$\frac{x^2}{4} + \frac{y^2}{3} = 1$$

$$\therefore \quad y = \frac{\sqrt{3}}{2}\sqrt{4-h^2} \text{ at } x = h$$

Let $R(x_1, 0)$.

PQ is the chord of contact.

So, $$\frac{xx_1}{5} = 1$$

or $$x = \frac{4}{x_1}$$

which is the equation of PQ. At $x = h$,

$$\frac{a}{a} = h \text{ or } x_1 = \frac{4}{h}$$

$$\Delta(h) = \text{Area of } \Delta PQR = \frac{1}{2} \times PQ \times RT$$

$$= \frac{1}{2} \times \frac{2\sqrt{3}}{2}\sqrt{4-h^2} \times (x_1 - h)$$

$$= \frac{\sqrt{3}}{2h}(4-h^2)^{3/2}$$

$$\Delta'(h) = \frac{-\sqrt{3}(4+2h^2)}{2h^2}\sqrt{4-h^2} < 0$$

Therefore, $\Delta(h)$ is always decreasing.

So, $\Delta_1 = \text{Maximum of } \Delta(h) = \frac{45\sqrt{5}}{8}$ at $h = \frac{1}{2}$

and $\Delta_2 = \text{Maximum of } \Delta(h) = \frac{9}{2}$ at $h = 1$

So, $$\frac{8}{\sqrt{5}}\Delta_1 - 8\Delta_2$$

$$= \frac{8}{\sqrt{5}} \times \frac{45\sqrt{5}}{8} - 8 \times \frac{9}{2}$$

$$= 45 - 36 = 9$$

22. ***(b)*** Let $Q(x_2, y_2, z_2)$ be the image of $A(2, 1, 6)$ about plane

$x + y - 2z = 3$. Then,

$$\frac{x_2 - 2}{1} = \frac{y_2 - 1}{1} = \frac{z_2 - 6}{-2} = \frac{-2(2+1-12-3)}{1^2+1^2+2^2} = 4$$

$$\Rightarrow \quad (x_2, y_2, z_2) \equiv (6, 5, -2)$$

23. ***(c)*** Let $\frac{x-2}{3} = \frac{y-1}{4} = \frac{z-6}{5} = \lambda$

$x = 2 + 3\lambda, y = 1 + 4\lambda, z = 6 + 5\lambda$ lies on plane

$x + y - 2z = 3$

$\Rightarrow$ $2 + 3\lambda + 1 + 4\lambda - 2(6 + 5\lambda) = 3$

or $3 + 7\lambda - 12 - 10\lambda = 3$

or $-3\lambda = 12$ or $\lambda = -4$

Point $B \equiv (-10, -15, -14)$

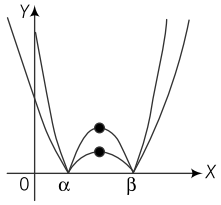
Sol. (Q. Nos. 24-25)

24. ***(b)*** **25.** ***(a)***

$$|px^2 + qx + r| \le |Px^2 + Qx + R|, \forall x \in R \quad \ldots(i)$$

From the graph we can see that this is possible only when both equations have same roots.

Thus, α and β are roots of $Px^2 + Qx + R = 0$ and also of $px^2 + qx + r = 0$.



So, from Eq. (i),

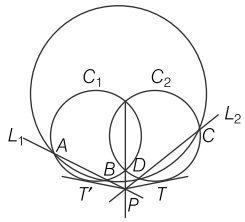
$$|p||x-\alpha||x-\beta| \le |P||x-\alpha||x-\beta| \Rightarrow |p| \le |P|$$

Also, $$\left|\frac{4pr - q^2}{4p}\right| \le \left|\frac{4PR - Q^2}{4P}\right|$$

$$\Rightarrow \quad |d| \le \left|\frac{p}{P}\right||D| \Rightarrow |d| \le |D| \qquad [\because |p| \le |P|]$$

Sol. (Q. Nos. 26-27)

26. ***(d)*** **27.** ***(a)***



The given circles are

$$C_1 : x^2 + y^2 - 4 = 0 \quad \ldots(i)$$

and $$C_2 : x^2 + y^2 - 4x - 2y + 1 = 0 \quad \ldots(ii)$$

The given lines are

$$L_1 : x + 2y + a = 0 \text{ and } L_2 : 12x - 6y - 41 = 0$$

If A, B, C and D are concyclic, then

$$PA \cdot PB = PC \cdot PD \text{ or } PT^2 = PT'^2$$

Therefore, point P will lie on the radical axis of both the circles C_1 and C_2, i.e. $L_3 : 4x + 2y - 5 = 0$.

Now, lines L_1, L_2 and L_3 are concurrent. Therefore,

$$\begin{vmatrix} 4 & 2 & -5 \\ 1 & 2 & a \\ 12 & -6 & -41 \end{vmatrix} = 0 \text{ or } a = 2$$

Also, for this value of a, the line meets circle (i) at two distinct points.

The family of circles through C_1 and L_1 is

$$x^2 + y^2 - 4 + \lambda(x + 2y + 2) = 0$$

Common chord with circle C_2 is

$$(\lambda + 4)x + 2(\lambda + 1)y + 2\lambda - 5 = 0$$

which is the same as $L_2 : 12x - 6y - 41 = 0$

Therefore, $$\lambda = -\frac{8}{5}$$

Hence, the equation of circle is

$$5x^2 + 5y^2 - 8x - 16y - 36 = 0$$

28. **A → p, q; B → p, q, r; C → q, s; D → s**

(A) $I = \int_{-2}^{2} (\alpha x^3 + \beta x + \gamma)\,dx$

$\alpha x^3 + \beta x$ is an odd function.

$I = 0 + 2\int_0^2 \gamma\,dx = 2 \times 2\gamma = 4\gamma$

(B) $I = \frac{1}{2}\int_0^1 2\sin\alpha x \sin\beta x\,dx$

$I = \frac{1}{2}\int_0^1 [\cos(\alpha - \beta)x - \cos(\alpha + \beta)x]\,dx$

$= \frac{1}{2}\left[\frac{\sin(\alpha - \beta)x}{\alpha - \beta} - \frac{\sin(\alpha + \beta)x}{\alpha + \beta}\right]_0^1$

$= \frac{1}{2}\left[\frac{\sin(\alpha - \beta)}{\alpha - \beta} - \frac{\sin(\alpha + \beta)}{\alpha + \beta}\right]$

Also, $2\alpha = \tan\alpha$ and $2\beta = \tan\beta$

$\therefore \quad 2(\alpha - \beta) = \tan\alpha - \tan\beta$

and $\quad 2(\alpha + \beta) = \tan\alpha + \tan\beta$

$2(\alpha - \beta) = \frac{\sin(\alpha - \beta)}{\cos\alpha\cos\beta}$ and $2(\alpha + \beta) = \frac{\sin(\alpha + \beta)}{\cos\alpha\cos\beta}$

Substituting these values, we get

$I = (\cos\alpha\cos\beta) - (\cos\alpha\cos\beta) = 0$

(C) $f(x + \alpha) + f(x) = 0$ or $f(x + 2\alpha) + f(x + \alpha) = 0$

or $\quad f(x + 2\alpha) = f(x)$

Thus, $f(x)$ is periodic with period 2α.

Hence,

$$\int_\beta^{\beta + 2\gamma\alpha} (\alpha x^3 + \beta x + \gamma)\,dx = \gamma\int_0^{2\alpha} f(x)\,dx$$

(D) Let $I = \int_0^\alpha [\sin x]\,dx, \alpha \in [(2\beta + 1)\pi, (2\beta + 2)\pi], \beta \in N$

[where [.] denotes the greatest integer function]

$= \int_0^{2\beta\pi} [\sin x]\,dx + \int_{2\beta\pi}^{(2\beta + 1)\pi} [\sin x]\,dx + \int_{(2\beta + 1)\pi}^{\alpha} [\sin x]\,dx$

$= \beta\int_0^{2\pi} [\sin x]\,dx + 0 + \int_{(2\beta + 1)\pi}^{\alpha} (-1)\,dx$

$= -\beta\pi + (2\beta + 1)\pi - \alpha = (\beta + 1)\pi - \alpha$

Thus, $\gamma\int_0^\alpha [\sin x]\,dx$ depends on α, β and γ.

29. **A → q; B → r; C → p; D → r**

(A) Since, a, b, c are in GP.

Hence, $\quad b^2 = ac$

$\Rightarrow \quad 2\log_{10} b = \log_{10} a + \log_{10} c$

$\Rightarrow \quad \frac{2}{\log_b 10} = \frac{1}{\log_a 10} + \frac{1}{\log_c 10}$

$\Rightarrow \quad \frac{2}{y} = \frac{1}{x} + \frac{1}{z}$

Hence, x, y, z are in HP.

(B) $\quad \frac{a + be^x}{a - be^x} = \frac{b + ce^x}{b - ce^x} = \frac{c + de^x}{c - de^x}$

$\Rightarrow \quad \frac{2a}{a - be^x} - 1 = \frac{2b}{b - ce^x} - 1 = \frac{2c}{c - de^x} - 1$

$\Rightarrow \quad \frac{a - be^x}{a} = \frac{b - ce^x}{b} = \frac{c - de^x}{c}$

$\Rightarrow \quad 1 - \frac{b}{a}e^x = 1 - \frac{c}{b}e^x = 1 - \frac{d}{c}e^x$

$\Rightarrow \quad \frac{b}{a} = \frac{c}{b} = \frac{d}{c}$

Hence, a, b, c, d are in GP.

(C) Given, $2b = a + c, x^2 = ab, y^2 = bc$

Now, $\quad x^2 + y^2 = b(a + c)$

$= b(2b) = 2b^2$

$\Rightarrow \quad x^2 + y^2 = 2b^2$

Hence, x^2, b^2, y^2 are in AP.

(D) $x \log a = y \log b = z \log c = k$ (say)

Also, $\quad y^2 = xz$

$\Rightarrow \quad \frac{k^2}{(\log b)^2} = \frac{k^2}{\log a \log c}$

Hence, $\log a, \log b, \log c$ are in GP.

30. **A → p, r, s; B → q, s; C → q, s; D → p, r, s**

We have, $f(x) = \frac{x^2 - 6x + 5}{x^2 - 5x + 6} = \frac{(x - 5)(x - 1)}{(x - 2)(x - 3)}$

(A) If $-1 < x < 1$, then $f(x) = \frac{(-\text{ve})(-\text{ve})}{(-\text{ve})(-\text{ve})} = +\text{ve}$

$\therefore f(x) > 0$

Also, $f(x) - 1 = \frac{-x - 1}{x^2 - 5x + 6} = \frac{-(x + 1)}{(x - 2)(x - 3)}$

For $-1 < x < 1$, $f(x) - 1 = \frac{-(+\text{ve})}{(-\text{ve})(-\text{ve})} = -\text{ve}$

or $f(x) - 1 < 0$ or $f(x) < 1$

$\therefore 0 < f(x) < 1$

(B) If $1 < x < 2$, then $f(x) = \frac{(-\text{ve})(+\text{ve})}{(-\text{ve})(-\text{ve})} = -\text{ve}$

Therefore, $f(x) < 0$ and so $f(x) < 1$

(C) If $3 < x < 5$, then

$$f(x) = \frac{(-\text{ve})(+\text{ve})}{(+\text{ve})(+\text{ve})} = -\text{ve}$$

Therefore, $f(x) < 0$ and so $f(x) < 1$

(D) For $x > 5$, $f(x) > 0$

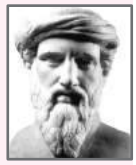
Also, $f(x) - 1 = \frac{-(x + 1)}{(x - 2)(x - 5)} < 0$ for $x > 5$

or $\quad f(x) < 1$

$\therefore \quad 0 < f(x) < 1$

Do You Know

Calculus means Pebbles in Greek



The famous Greek Mathematician, Pythagoras, used to have small rocks to signify numbers while working on mathematical equations. This led to origin of word calculus which means Pebbles in Greek.

BRAiN TEASERS

To boost up your rank in JEE Main and Advanced

1. Prove that there does not exist any positive integers $n < 2310$ such that $n(2310 - n)$ is a multiple of 2310.

2. There are four basketball players A, B, C, D. Initially, the ball is with A. The ball is always passed from one person to a different person. In how many ways can the ball come back to A after seven passes?
(e.g. $A \to C \to B \to D \to A \to B \to C \to A$ and $A \to D \to A \to D \to C \to A \to B \to A$ are two ways in which the ball can come back to A after seven passes)

3. For any natural number $n > 1$, write the infinite decimal expansion of $1/n$ (i.e. we write $1/2 = 0.4\bar{9}$ as its infinite decimal expansion, not 0.5). Determine the length of the non-periodic part of the (infinite) decimal expansion of $1/n$.

4. Solve approximately the equation $\sin\theta = 0.52$

5. Find the number of positive integers which divide 10^{999} but not 10^{998}.

6. If $x^2 + y^2 = c^2$, then find the least value of $\frac{1}{x^2} + \frac{1}{y^2}$.

7. If $x + y + z = xyz$, prove that

$$\frac{2x}{1-x^2} + \frac{2y}{1-y^2} + \frac{2z}{1-z^2} = \frac{2x}{1-x^2} \cdot \frac{2y}{1-y^2} \cdot \frac{2z}{1-z^2}$$

8. Show that $3^{2008} + 4^{2009}$ can be written as product of two positive integers each of which is larger than $(2009)^{182}$.

9. Find the values of a, b, c which will make each of the expressions $x^4 + ax^3 + bx^2 + cx + 1$ and $x^4 + 2ax^3 + 2bx^2 + 2cx + 1$ a perfect square.

10. Find the cube root of 126 to 5 places of decimal.

Solutions

1. As, $n(2310 - n)$ is a multiple of 2310.

$\therefore \quad n(2310 - n) = 2310k$ [where k is some integer]

$\Rightarrow \quad 2310n - n^2 = 2310k$

$\Rightarrow \quad 2310n - 2310k = n^2$

$\Rightarrow \quad 2310(n - k) = n^2$

$\Rightarrow \quad n - k = \frac{n^2}{2310}$

As, $n - k$ is an integer, so $\frac{n^2}{2310}$ is also an integer.

So, n^2 is divisible by 2310.

As, $2310 = 2 \times 3 \times 5 \times 7 \times 11$

Thus, n is also divisible by 2310.

But n is less than 2310, i.e. $n < 2310$. So, we can say that n is not divisible by 2310.

2. Let P_i denotes ith pass, then

$1 \xrightarrow{P_1} 2 \xrightarrow{P_2} 3 \xrightarrow{P_3} 4 \xrightarrow{P_4} 5 \xrightarrow{P_5} 6 \xrightarrow{P_6} 7 \xrightarrow{P_7} 8$

***Case* I** If in between A gets 2 passes

$1. \to A \times A \times A \times\times A \to 3 \cdot 1 \cdot 3 \cdot 1 \cdot 3 \cdot 2 \cdot 1 = 54$

$2. \to A \times A \times\times A \times A \to 3 \cdot 1 \cdot 3 \cdot 2 \cdot 1 \cdot 3 \cdot 1 = 54$

$3. \to A \times\times A \times A \times A \to 3 \cdot 2 \cdot 1 \cdot 3 \cdot 1 \cdot 3 \cdot 1 = 54$

***Case* II** If in between A gets 1 pass

$4. \to A \times A \times\times\times\times A \to 3 \cdot 1 \cdot 3 \cdot 2 \cdot 2 \cdot 2 \cdot 1 = 72$

$5. \to A \times\times A \times\times\times A \to 3 \cdot 2 \cdot 1 \cdot 3 \cdot 2 \cdot 2 \cdot 1 = 72$

$6. \to A \times\times\times A \times\times A \to 3 \cdot 2 \cdot 2 \cdot 1 \cdot 3 \cdot 2 \cdot 1 = 72$

$7. \to A \times\times\times\times A \times A \to 3 \cdot 2 \cdot 2 \cdot 2 \cdot 1 \cdot 3 \cdot 1 = 72$

***Case* III** If in between A gets no pass

$8. \to A \times\times\times\times\times\times A \to 3 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 1 = 96$

Hence, total number of ways

$= 3 \times 54 + 4 \times 72 + 96$

$= 546$

3. $\frac{1}{n} = 0 \cdot a_1 a_2 a_3 \ldots a_k b_1 b_2 b_3 \ldots b_r$

where, r is length of repeating part.

$$\Rightarrow \quad \frac{1}{n} = \frac{a_1 a_2 \ldots a_k}{10^k} + \frac{b_1 b_2 \ldots b_r}{10^{k+r}}$$

$$= \frac{a_1 a_2 \ldots a_k}{10^k} + \frac{b_1 b_2 \ldots b_r}{10^k}\left(\frac{1}{10^r - 1}\right)$$

$$= \frac{(a_1 a_2 \ldots a_k)(10^r - 1) + b_1 b_2 \ldots b_r}{10^k (10^r - 1)}$$

Hence, $10^k = 2^k \cdot 5^k \Rightarrow k$ is higher of the two exponents to 2, 5 in prime factorisation of n.

$\Rightarrow k$ = maximum (exponent of 2, exponent of 5).

4. Since, when $\sin\theta$ is very nearly equal to $\frac{1}{2}$, θ must be nearly equal to $\frac{\pi}{6}$. Let $\theta = \frac{\pi}{6} + x$, where x is small.

$$\therefore \ 0.52 = \sin\left(\frac{\pi}{6} + x\right)$$

$$= \sin\frac{\pi}{6}\cos x + \cos\frac{\pi}{6}\sin x = \frac{1}{2}\cos x + \frac{\sqrt{3}}{2}\sin x$$

Since, x is very small, we have

$\cos x = 1$ and $\sin x = x$ (nearly)

$$\therefore \ 0.52 = \frac{1}{2} + \frac{\sqrt{3}}{2}x$$

$$\therefore \quad x = 0.02 \times \frac{2}{\sqrt{3}} \text{ rad} = 1.32 \text{ rad} = 31°19' \text{ (nearly)}$$

5. All the positive divisors of $p^r (p$ is positive prime, $r \in N)$ are $1, p, p^2, p^3, \ldots, p^r$. That is the number of positive divisors of p^r is $r + 1$. Similarly, the number of the positive divisors of q^s (q is a positive prime, $s \in N$) is $s + 1$, so the number of positive divisors of $p^r q^s$ is $(r + 1)(s + 1)$.

Now, $10^{999} = 2^{999} \times 5^{999}$, where 2 and 5 are positive primes. So, the number of positive divisors of 10^{999} is $(999 + 1)(999 + 1) = 1000^2$. For similar reasons, the number of positive divisors of 10^{998} is 999^2. So, the number of positive numbers which divides 10^{999} but not 10^{998} is $1000^2 - 999^2 = (1000 + 999)(1000 - 999)$

$$= 1999 \times 1 = 1999$$

6. Let $z' = \frac{1}{x^2} + \frac{1}{y^2} = \frac{y^2 + x^2}{x^2 y^2} = \frac{c^2}{x^2 y^2}$

$\therefore z'$ will be minimum when $\frac{x^2 y^2}{c^2}$ will be maximum.

Now, let $z = \frac{x^2 y^2}{c^2} = \frac{1}{c^2}(x^2)(y^2)$

$\therefore z$ will be maximum when $x^2 y^2$ is maximum but $(x^2)(y^2)$ is the product of two factors whose sum is $x^2 + y^2 = c^2$.

$\therefore x^2 y^2$ will be maximum when both these factors are equal, i.e. when $\frac{x^2}{1} = \frac{y^2}{1} = \frac{x^2 + y^2}{1} = \frac{c^2}{1}$

$\therefore$ Maximum value of $z = \frac{c^2}{4}$

$\therefore$ Least value of $\frac{1}{x^2} + \frac{1}{y^2} = \frac{4}{c^2}$

7. Put $x = \tan A$, $y = \tan B$ and $z = \tan C$, so that we have

$\tan A + \tan B + \tan C = \tan A \tan B \tan C$

$$\therefore \quad \frac{\tan A + \tan B}{1 - \tan A \tan B} = -\tan C$$

$$\Rightarrow \quad \tan(A + B) = \tan(\pi - C)$$

$$\Rightarrow \quad A + B + C = n\pi + \pi$$

$$\therefore \ \frac{2x}{1 - x^2} + \frac{2y}{1 - y^2} + \frac{2z}{1 - z^2}$$

$$= \frac{2\tan A}{1 - \tan^2 A} + \frac{2\tan B}{1 - \tan^2 B} + \frac{2\tan C}{1 - \tan^2 C}$$

$$= \tan 2A + \tan 2B + \tan 2C$$

$$= \tan 2A \tan 2B \tan 2C$$

$$= \frac{2x}{1 - x^2} \cdot \frac{2y}{1 - y^2} \cdot \frac{2z}{1 - z^2}$$ **Hence proved.**

8. We know that,

$$x^4 + 4y^4 = (x^2 + 2xy + 2y^2)(x^2 - 2xy + 2y^2)$$

We observe that for any integers x, y

$$x^2 + 2xy + 2y^2 = (x + y)^2 + y^2 \geq y^2$$

and $\quad x^2 - 2xy + 2y^2 = (x - y)^2 + y^2 \geq y^2$

We write $\quad 3^{2008} + 4^{2009} = 3^{2008} + 4 \cdot (4^{2008})$

$$= (3^{502})^4 + 4 \cdot (4^{502})^4$$

Taking $x = 3^{502}$ and $y = 4^{502}$, we see that

$3^{2008} + 4^{2009} = ab$, where $a \geq (4^{502})^2$, $b \geq (4^{502})^2$

But we have,

$$(4^{502})^2 = 2^{2008} > 2^{2002} = (2^{11})^{182} > (2009)^{182}$$

$$[\because 2^{11} = 2048 > 2009]$$

9. Suppose

$$x^4 + ax^3 + bx^2 + cx + 1 \equiv \left(x^2 + \frac{ax}{2} + 1\right)^2 \quad \ldots(i)$$

and $x^4 + 2ax^3 + 2bx^2 + 2cx + 1 \equiv (x^2 + ax + 1)^2 \ldots(ii)$

Equating like coefficients from Eqs. (i) and (ii), we get

$b = \frac{a^2}{4} + 2$ and $c = a$ and $2b = a^2 + 2, 2c = 2a$

$$\therefore \quad a^2 + 2 = 2\left(\frac{a^2}{4} + 2\right) \Rightarrow a^2 + 2 = \frac{a^2}{2} + 4$$

$$\Rightarrow \quad \frac{a^2}{2} = 2 \Rightarrow a^2 = 4 \Rightarrow a = \pm 2$$

$$\therefore \quad a = \pm 2, c = \pm 2, b = 3$$

10. We have,

$$(126)^{1/3} = (5^3 + 1)^{1/3} = 5\left(1 + \frac{1}{5^3}\right)^{1/3}$$

$$= 5\left(1 + \frac{1}{3}\cdot\frac{1}{5^3} - \frac{1}{9}\cdot\frac{1}{5^6} + \frac{5}{81}\cdot\frac{1}{5^9} - \ldots\right)$$

$$= 5 + \frac{1}{3 \cdot 5^2} - \frac{1}{9 \cdot 5^5} + \frac{1}{81 \cdot 5^7} - \ldots$$

$$= 5 + \frac{1}{3}\cdot\frac{2^2}{10^2} - \frac{1}{9}\cdot\frac{2^5}{10^5} + \frac{1}{81}\cdot\frac{2^7}{10^7} - \ldots$$

$$= 5 + \frac{0.04}{3} - \frac{0.00032}{9} + \frac{0.0000128}{81} - \ldots$$

$$= 5 + 0.013333 \ldots - 0.000035 \ldots + \ldots$$

$$= 5.01329, \text{ to five places of decimals.}$$

TARGET CBSE 2017

TEST DRIVE

Comprehensive Simulator Test Series for JEE Main & Advanced

BOARD EXAM

(PAPER ONE PREP UP)

FULL TEST 1

(XII SYLLABUS)
WITH SOLUTIONS

Typical Questions with Crispy Solutions

INSTRUCTIONS

- Question Numbers 1-4 in **Section A** carrying 1 mark each.
- Question Numbers 5-12 in **Section B** carrying 2 marks each.
- Question Numbers 13-23 in **Section C** carrying 4 marks each.
- Question Numbers 24-29 in **Section D** carrying 6 marks each.

Section A

1. Is R defined on the set $A = \{1, 2, 3, 4, 5, 6\}$ as $R = \{(x, y) : y \text{ is divisible by } x\}$ symmetric?

2. Calculate the direction cosines of the vector $\mathbf{a} = 3\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 5\hat{\mathbf{k}}$.

3. For what value of x, the matrix $\begin{bmatrix} 3-x & x \\ 4-x & x+1 \end{bmatrix}$ is non-invertible?

4. If $\tan^{-1} x + \tan^{-1} y = \frac{\pi}{4}$; $xy < 1$, then write the value of $x + y + xy$. [AI 2014]

Section B

5. If A and B are symmetric matrices of the same order, then prove that $AB + BA$ is symmetric.

6. Differentiate $y = \tan^{-1}\left(\sqrt{\frac{1+\cos x}{1-\cos x}}\right)$, $0 < x < \pi$.

7. If $x^y = e^{x-y}$, then prove that $\frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$.

8. Solve the differential equation $\frac{dy}{dx} + \sqrt{\frac{1-y^2}{1-x^2}} = 0$.

9. Show that the vectors $\mathbf{a} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$, $\mathbf{b} = 3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 3\hat{\mathbf{k}}$ and $\mathbf{c} = 5\hat{\mathbf{i}} + 5\hat{\mathbf{j}} + \hat{\mathbf{k}}$ form a right angled triangle.

10. A problem in Mathematics is given to 3 students whose chances of solving it are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$. What is the probability that the problem is solved?

11. The radius of a circle is increasing uniformly at the rate of 5 cm/s. Find the rate at which the area of the circle is increasing, when the radius is 3 cm.

12. Show that the tangents to the curve $y = x^2 - 5x + 6$ at the points (2, 0) and (3, 0) are at right angles.

Section C

13. Prove the following

$$\cot^{-1}\left(\frac{xy+1}{x-y}\right) + \cot^{-1}\left(\frac{yz+1}{y-z}\right) + \cot^{-1}\left(\frac{zx+1}{z-x}\right) = 0;$$

$(0 < xy, yz, zx < 1)$ [AI 2015]

14. Using properties of determinants, prove that

$$\begin{vmatrix} a+x & y & z \\ x & a+y & z \\ x & y & a+z \end{vmatrix} = a^2(a+x+y+z)$$

[AI 2014]

15. Bag A contains 4 black and 6 red balls and bag B contains 7 black and 3 red balls. A die is thrown. If 1 or 2 appears on it, then bag A is chosen, otherwise bag B. If two balls are drawn at random (without replacement) from the selected bag, then find the probability of one of them being red and another black.

16. Find the value of a for which the function f defined as

$$f(x) = \begin{cases} a\sin\frac{\pi}{2}(x+1), & x \le 0 \\ \frac{\tan x - \sin x}{x^3}, & x > 0 \end{cases}$$ is continuous at $x = 0$.

[Delhi 2011]

17. Using vectors, find the area of triangle with vertices $A(1, -1, 2)$, $B(2, 1, -1)$ and $C(3, -1, 2)$.

Or

Let $\mathbf{a} = 2\hat{\mathbf{i}} + \hat{\mathbf{k}}$, $\mathbf{b} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{c} = 4\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 7\hat{\mathbf{k}}$ be three vectors. Find a vector $\mathbf{r}$ which satisfies $\mathbf{r} \times \mathbf{b} = \mathbf{c} \times \mathbf{b}$ and $\mathbf{r} \cdot \mathbf{a} = 12$.

18. There are 40 hardworking scholars in a class. Out of which 10 are sports persons. Three scholars are selected at random out of them . Write the probability distribution for selected persons who are sports persons. Find the mean of distribution. Explain the importance of sports in education.

19. Find the vector and cartesian equation of the line passing through the point $(-1, 2, 3)$ and perpendicular to the lines $\frac{x}{2} = \frac{y-1}{-3} = \frac{z+2}{-2}$ and $\frac{x+3}{-1} = \frac{y+3}{2} = \frac{z-1}{3}$.

20. Evaluate $\int_{-\pi}^{\pi} \frac{2x(1+\sin x)}{1+\cos^2 x}\,dx$.

21. Find the particular solution of differential equation $2xy + y^2 - 2x^2\frac{dy}{dx} = 0$, $y(1) = 2$.

Or

Form the differential equation of all circles touching the Y-axis at the origin.

22. Consider the binary operations $* : R \times R \to R$ and $o = R \times R \to R$ defined as $a * b = |a - b|$ and $a\ o\ b = a$. For all $a, b \in R$. Show that $*$ is commutative but not associative, 'o' is associative but not commutative.

[AI 2012]

23. Evaluate $\int \frac{\log|x|}{(x+1)^2}\,dx$.

[AI 2015]

Section D

24. If $A = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 3 \\ 0 & -2 & 1 \end{bmatrix}$, then find A^{-1}. Using A^{-1} solve the system of linear equations

$x - 2y = 10$, $2x + y + 3z = 8$, $-2y + z = 7$

Or

The sum of three numbers is 6. If we multiply the third number by 2 and add first number to it we get 7. By adding second and third numbers to three times the first number, we get 12.

Represent the following information mathematically and solve using matrices.

25. A factory manufactures two types of machines A and B. Each type is made of certain metal. The factory has only 480 kg of this metal available in a day. To manufacture machine A, 10 kg of metal is required and 20 kg is required for B. Machine A and B require 15 and 10 min to be painted. Painting department can use only 400 min in a day. The factory earns profit of ₹ 10500 on machine A and ₹ 9000 on machine B. State as a LPP and maximize the profit.

26. Using integration, find the area of the region bounded by the curves $y = \sqrt{5 - x^2}$ and $y = |x - 1|$.

27. Evaluate the following

$$\int_0^{\pi/2} \frac{x\sin x\cos x}{\sin^4 x + \cos^4 x}\,dx.$$

Or

Evaluate $\int_0^4 (x + e^{2x})\,dx$ as the limit of a sum.

28. Find the distance of the point $P(-1, -5, -10)$ from the point of intersection of the line $\mathbf{r} = (2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}}) + \lambda(3\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 2\hat{\mathbf{k}})$ and the plane $\mathbf{r} \cdot (\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}) = 5$. Also, find the image of the point in the plane.

29. Find the intervals in which

$$f(x) = 2\log(x - 2) - x^2 + 4x + 1$$

is increasing or decreasing.

Shake your Mind

In the diagram the numbers from 1 to 9 are arranged in a square in such a way that the number in the second row is twice that in the first row and the number in the bottom row three times that in the top row.

I am told that there are three other ways of arranging the numbers so as to produce the same result.

Can you find the other three ways?

1	9	2
3	8	4
5	7	6

Answers with Explanation

1. No, as $(2, 4) \in R$ but $(4, 2) \notin R$.

2. $|\mathbf{a}| = \sqrt{(3)^2 + (-2)^2 + (5)^2} = \sqrt{38}$

$\therefore l = \frac{3}{\sqrt{38}}, m = -\frac{2}{\sqrt{38}}$ and $n = \frac{5}{\sqrt{38}}$

3. The given matrix is non-invertible, if $|A| = 0$

$$\therefore \quad \begin{vmatrix} 3-x & x \\ 4-x & x+1 \end{vmatrix} = 0$$

$$\Rightarrow (3-x)(x+1) - x(4-x) = 0 \quad \Rightarrow \quad x = \frac{3}{2}$$

4. Given, $\tan^{-1} x + \tan^{-1} y = \frac{\pi}{4}, xy < 1$

$$\tan^{-1}\left(\frac{x+y}{1-xy}\right) = \frac{\pi}{4} \quad \Rightarrow \quad \frac{x+y}{1-xy} = 1$$

$$\Rightarrow \quad x + y = 1 - xy \quad \Rightarrow \quad x + y + xy = 1$$

5. $(AB + BA)^T = (AB)^T + (BA)^T = B^T A^T + A^T B^T$

$= BA + AB \qquad [\because B^T = B, A^T = A]$

$= AB + BA$

$\therefore (AB + BA)$ is symmetric.

6. $y = \tan^{-1}\left(\sqrt{\frac{1+\cos x}{1-\cos x}}\right)$

$$= \tan^{-1}\left(\sqrt{\frac{2\cos^2\frac{x}{2}}{2\sin^2\frac{x}{2}}}\right) = \tan^{-1}\left|\cot\frac{x}{2}\right|$$

$$= \tan^{-1}\left\{\tan\left(\frac{\pi}{2} - \frac{x}{2}\right)\right\} = \frac{\pi}{2} - \frac{x}{2}$$

$$\therefore \quad \frac{dy}{dx} = 0 - \frac{1}{2} = -\frac{1}{2}$$

7. $x^y = e^{x-y}$

On taking logarithm both sides, we get

$$y \log x = x - y \quad \Rightarrow \quad y(\log x + 1) = x$$

$$\Rightarrow \quad y = \frac{x}{1 + \log x}$$

$$\therefore \quad \frac{dy}{dx} = \frac{1 + \log x - 1}{(1 + \log x)^2} = \frac{\log x}{(1 + \log x)^2}$$ **Hence proved.**

8. $\frac{dy}{dx} + \sqrt{\frac{1-y^2}{1-x^2}} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$

$$\Rightarrow \quad \frac{dy}{\sqrt{1-y^2}} = -\frac{dx}{\sqrt{1-x^2}}$$

$$\Rightarrow \quad \int \frac{dy}{\sqrt{1-y^2}} = -\int \frac{dx}{\sqrt{1-x^2}}$$

$$\Rightarrow \quad \sin^{-1} y = -\sin^{-1} x + \sin^{-1} C$$

$$\Rightarrow \quad \sin^{-1} y + \sin^{-1} x = \sin^{-1} C$$

$$\Rightarrow \sin^{-1}[y\sqrt{1-x^2} + x\sqrt{1-y^2}] = \sin^{-1} C$$

$$\Rightarrow \quad y\sqrt{1-x^2} + x\sqrt{1-y^2} = C$$

9. We have, $\mathbf{a} + \mathbf{b} = \mathbf{c}$

Let $\mathbf{AC} = \mathbf{c}$, $\mathbf{CB} = \mathbf{a}$

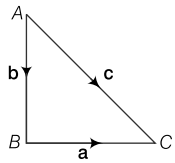
and $\mathbf{AB} = \mathbf{b}$, then $\mathbf{AB} + \mathbf{BC} = \mathbf{AC}$

Hence, vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ form the sides of triangle.

Also, $\mathbf{a} \cdot \mathbf{b} = 6 + 6 - 12 = 0$

$\therefore$ **AB** is perpendicular to **CB**.

Hence, ΔABC is a right angled triangle. **Hence proved.**



10. Let A, B and C be the respective events of solving the problem by students A, B and C, respectively. Then,

$$P(A) = \frac{1}{2}, P(B) = \frac{1}{3} \text{ and } P(C) = \frac{1}{4}$$

$\therefore$ Required probability $= P(A \cup B \cup C)$

$= 1 - P(A \cup B \cup C) = 1 - P(\bar{A} \cap \bar{B} \cap \bar{C})$

$= 1 - P(\bar{A})\, P(\bar{B})\, P(\bar{C})$

$$= 1 - \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{4}\right) = \frac{3}{4}$$

11. Let r be the radius and A be the area of a circle at any time t. Then, $A = \pi r^2$

On differentiating both sides w.r.t. t, we get

$$\frac{dA}{dt} = 2\pi r \times \frac{dr}{dt} \quad \Rightarrow \quad \frac{dA}{dt} = 2\pi r \times 5 \quad \left[\because \frac{dr}{dt} = 5 \text{ cm/s}\right]$$

$$\Rightarrow \quad \left(\frac{dA}{dt}\right)_{r=3} = 2\pi \times 3 \times 5 \text{ cm}^2/\text{s} = 30\pi \text{ cm}^2/\text{s}$$

12. $y = x^2 - 5x + 6$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = 2x - 5$$

$$m_1 = \left(\frac{dy}{dx}\right)_{(2,0)} = 2 \times 2 - 5 = -1$$

and $m_2 = \left(\frac{dy}{dx}\right)_{(3,0)} = 2 \times 3 - 5 = 1$

Clearly, $m_1 \times m_2 = -1 \times 1 = -1$

Thus, the tangents to the given curve at $(2, 0)$ and $(3, 0)$ and at right angles.

13. LHS $= \tan^{-1}\left(\frac{x-y}{1+xy}\right) + \tan^{-1}\left(\frac{y-z}{1+yz}\right) + \tan^{-1}\left(\frac{z-x}{1+zx}\right)$

$= (\tan^{-1} x - \tan^{-1} y) + (\tan^{-1} y - \tan^{-1} z)$

$\qquad + (\tan^{-1} z - \tan^{-1} x)$

$= 0 =$ RHS **Hence proved.**

14. LHS $= \begin{vmatrix} a+x & y & z \\ x & a+y & z \\ x & y & a+z \end{vmatrix}$

On applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\begin{vmatrix} a+x+y+z & y & z \\ a+x+y+z & a+y & z \\ a+x+y+z & y & a+z \end{vmatrix}$$

$$= (a + x + y + z)\begin{vmatrix} 1 & y & z \\ 1 & a+y & z \\ 1 & y & a+z \end{vmatrix}$$

On applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$$= (a + x + y + z)\begin{vmatrix} 1 & y & z \\ 0 & a & 0 \\ 0 & 0 & a \end{vmatrix}$$

$$= (a + x + y + z)[1(a^2 - 0)] = a^2(a + x + y + z)$$

= RHS **Hence proved.**

15. Let E, F and A be three events such that

E = selection of bag A

F = selection of bag B

A = getting one red and one black ball

Here, $P(E)$ = (getting 1 or 2 in a throw of die) $= \frac{2}{6} = \frac{1}{3}$

$$\therefore \quad P(F) = 1 - \frac{1}{3} = \frac{2}{3}$$

Also, $$P\left(\frac{A}{E}\right) = \frac{{}^6C_1 \times {}^4C_1}{{}^{10}C_2} = \frac{24}{45}$$

and $$P\left(\frac{A}{F}\right) = \frac{{}^3C_1 \times {}^7C_1}{{}^{10}C_2} = \frac{21}{45}$$

Now, by theorem of total probability

$$P(A) = P(E) \cdot P\left(\frac{A}{E}\right) + P(F) \cdot P\left(\frac{A}{F}\right)$$

$$= \frac{1}{3} \times \frac{24}{45} + \frac{2}{3} \times \frac{21}{45} = \frac{8}{45} + \frac{14}{45} = \frac{22}{45}$$

16. Since, $f(x)$ is continuous at $x = 0$.

$$\therefore \quad \text{LHL} = \text{RHL} = f(0)$$

Now, $$f(0) = a \sin\frac{\pi}{2}(0 + 1) = a$$

Again, $$\text{RHL} = \lim_{x \to 0^+} \frac{\tan x - \sin x}{x^3} = \lim_{h \to 0} \frac{\tan h - \sin h}{h^3}$$

$$= \lim_{h \to 0} \frac{\sin h(1 - \cos h)}{h^3 \cos h} = \lim_{h \to 0} \frac{\sin h \cdot 2\sin^2 h/2}{h^3 \cos h}$$

$$= 2\lim_{h \to 0} \frac{\sin h}{h} \cdot \lim_{h \to 0} \frac{\sin^2 h/2}{h^2/4} \times \frac{1}{4} \cdot \frac{1}{\cos h} = \frac{1}{2}$$

$$\therefore \quad a = \frac{1}{2}$$

17. Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be the position vectors of A, B and C, respectively.

Then, $\mathbf{AB} = \mathbf{b} - \mathbf{a} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 3\hat{\mathbf{k}}$ and $\mathbf{AC} = \mathbf{c} - \mathbf{a} = 2\hat{\mathbf{i}}$

$$\therefore \quad \mathbf{AB} \times \mathbf{AC} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 2 & -3 \\ 2 & 0 & 0 \end{vmatrix} = -6\hat{\mathbf{j}} - 4\hat{\mathbf{k}}$$

$$\therefore \quad |\mathbf{AB} \times \mathbf{AC}| = \sqrt{(-6)^2 + (-4)^2} = 2\sqrt{13}$$

So, area of $\Delta ABC = \frac{1}{2}|\mathbf{AB} \times \mathbf{AC}| = \frac{1}{2} \times 2 \times \sqrt{13} = \sqrt{13}$

Or

Given, $\mathbf{r} \times \mathbf{b} = \mathbf{c} \times \mathbf{b}$

$\Rightarrow \quad \mathbf{r} \times \mathbf{b} - \mathbf{c} \times \mathbf{b} = 0 \Rightarrow (\mathbf{r} - \mathbf{c}) \times \mathbf{b} = 0$

$\Rightarrow \quad \mathbf{r} - \mathbf{c}$ is parallel to $\mathbf{b}$.

$\Rightarrow \quad \mathbf{r} - \mathbf{c} = \lambda\mathbf{b}$ for some scalar λ

$\Rightarrow \quad \mathbf{r} = \mathbf{c} + \lambda\mathbf{b}$

Now, $\mathbf{r} \cdot \mathbf{a} = 12$

$\Rightarrow \quad (\mathbf{c} + \lambda\mathbf{b}) \cdot \mathbf{a} = 12 \Rightarrow (\mathbf{c} \cdot \mathbf{a}) + \lambda(\mathbf{b} \cdot \mathbf{a}) = 12$

$\Rightarrow \quad (8 + 7) + \lambda(2 + 1) = 12 \Rightarrow \lambda = -1$

$\therefore \quad \mathbf{r} = \mathbf{c} - \mathbf{b} = 3\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 6\hat{\mathbf{k}}$

18. Let X denotes the number of selected sports persons.

So, X can take values 0, 1, 2, 3.

$$P(X = 0) = \frac{{}^{30}C_3}{{}^{40}C_3} = \frac{406}{988}$$

$$P(X = 1) = \frac{{}^{30}C_2 \times {}^{10}C_1}{{}^{40}C_3} = \frac{435}{988}$$

$$P(X = 2) = \frac{{}^{30}C_1 \times {}^{10}C_2}{{}^{40}C_3} = \frac{135}{988}$$

$$P(X = 3) = \frac{{}^{10}C_3}{{}^{40}C_3} = \frac{12}{988}$$

Thus, probability distribution is given by

X	0	1	2	3
$P(X)$	$\frac{406}{988}$	$\frac{435}{988}$	$\frac{135}{988}$	$\frac{12}{988}$

Mean $= 1 \times \frac{435}{988} + 2 \times \frac{135}{988} + 3 \times \frac{12}{988} = \frac{741}{988}$

Importance of sports in education It takes care of mental and physical fitness of the body. It is helpful for the growth of student in the field of studies as well.

19. Let the direction ratios of the required line be a, b, c.

Since, it is perpendicular to the two given lines.

Therefore, $2a - 3b - 2c = 0$ ...(i)

and $-a + 2b + 3c = 0$ ...(ii)

On solving Eqs. (i) and (ii) by cross-multiplication, we get

$$\frac{a}{-5} = \frac{b}{-4} = \frac{c}{1} = k \text{ (say)}$$

Thus, the required line passes through $(-1, 2, 3)$ and has direction ratios proportional to $-5, -4, 1$.

So, its equation is $\frac{x+1}{-5} = \frac{y-2}{-4} = \frac{z-3}{1}$

The required line is parallel to the vector having direction ratios $-5, -4, 1$.

So, its vector equation is

$$\mathbf{r} = (-\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}) + \lambda(-5\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + \hat{\mathbf{k}})$$

20. $$I = \int_{-\pi}^{\pi} \frac{2x(1 + \sin x)}{1 + \cos^2 x}\,dx$$

$$\Rightarrow \quad I = \int_{-\pi}^{\pi} \frac{2x}{1 + \cos^2 x}\,dx + \int_{-\pi}^{\pi} \frac{2x \sin x}{1 + \cos^2 x}\,dx$$

Let $$I_1 = \int_{-\pi}^{\pi} \frac{2x}{1 + \cos^2 x}\,dx$$

Since, $\frac{2x}{1 + \cos^2 x}$ is an odd function.

$$\therefore \quad I_1 = 0$$

Let $$I_2 = \int_{-\pi}^{\pi} \frac{2x \sin x}{1 + \cos^2 x}\,dx$$

Since, $\frac{2x \sin x}{1 + \cos^2 x}$ is an even function.

$$\therefore \quad I_2 = 2\int_0^{\pi} \frac{2x \sin x}{1 + \cos^2 x}\,dx \quad \text{...(i)}$$

$$= 4\int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1 + \cos^2(\pi - x)}\,dx$$

$$= 4\int_0^{\pi} \frac{(\pi - x)\sin x}{1 + \cos^2 x}\,dx \quad \text{...(ii)}$$

On adding Eqs. (i) and (ii), we get

$$2I_2 = 4\pi\int_0^{\pi} \frac{1}{1+\cos^2 x}\sin x\, dx$$

$$\Rightarrow \quad 2I_2 = -4\pi\int_1^{-1} \frac{1}{1+t^2}\,dt \qquad [\text{where, } t = \cos x]$$

$$\Rightarrow \quad 2I_2 = -4\pi\,[\tan^{-1} t]_1^{-1}$$

$$\Rightarrow \quad 2I_2 = 2\pi^2 \Rightarrow I_2 = \pi^2$$

$$\therefore \quad I = 0 + \pi^2 = \pi^2$$

21. $\frac{dy}{dx} = \frac{2xy + y^2}{2x^2}$

Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$$v + x\frac{dv}{dx} = \frac{2v + v^2}{2} \Rightarrow 2v + 2x\frac{dv}{dx} = 2v + v^2$$

$$\Rightarrow \quad \frac{2}{v^2}dv = \frac{1}{x}dx \quad \Rightarrow \quad 2\int\frac{1}{v^2}dv = \int\frac{1}{x}dx$$

$$\Rightarrow \quad \frac{-2}{v} = \log|x| + C \quad \Rightarrow \quad \frac{-2x}{y} = \log|x| + C$$

When $x = 1$ and $y = 2$

$$-1 = 0 + C \Rightarrow C = -1$$

$$\therefore \quad \frac{-2x}{y} = \log|x| - 1 \Rightarrow y = \frac{2x}{1 - \log|x|}$$

Or

The equation of the family of circles touching Y-axis at the origin is

$(x-a)^2 + (y-0)^2 = a^2 \Rightarrow x^2 + y^2 - 2ax = 0$ ...(i)

where a is parameter.

On differentiating w.r.t. x, we get

$$2x + 2y\frac{dy}{dx} - 2a = 0 \Rightarrow a = x + y\frac{dy}{dx} \qquad ...(ii)$$

Putting the value of a from Eq. (ii) in Eq. (i), we get

$$x^2 + y^2 - 2x\left(x + y\frac{dy}{dx}\right) = 0 \Rightarrow y^2 - x^2 = 2xy\frac{dy}{dx}$$

22. Given, $* : R \times R \to R$ such that $a * b = |a - b|$

and $o : R \times R \to R$ such that $a\ o\ b = a$

Now, $a * b = |a - b|$

and $b * a = |b - a| = |-(a-b)| = |a - b|$

$\therefore \quad a * b = b * a$

Thus, * is commutative.

Again, $(a * b) * c = |a - b| * c = ||a - b| - c|$

and $a * (b * c) = a * |b - c| = |a - |b - c||$

$\therefore \quad (a * b) * c \neq a * (b * c)$

Thus, * is not associative.

Similarly, $a\ o\ b = a$ and $b\ o\ a = b$

$\therefore \quad a\ o\ b \neq b\ o\ a$

Thus, o is not commutative.

Again, $(a\ o\ b)\ o\ c = a\ o\ c = a$

and $a\,o(b\,o\,c) = a\,o\,b = a$

$\therefore \quad (a\ o\ b)\,o\,c = a\ o\ (b\ o\ c)$

Thus, o is associative.

23. Let $I = \int \frac{\log|x|}{(x+1)^2}dx = \int \underset{\text{I}}{\log|x|} \cdot \underset{\text{II}}{\frac{1}{(x+1)^2}}dx$

$$= \log|x| \cdot \left(\frac{-1}{x+1}\right) - \int \frac{-1}{x+1}\cdot\frac{1}{x}dx$$

$$= -\frac{\log|x|}{(x+1)} + \int\frac{1}{x(x+1)}dx$$

$$= -\frac{\log|x|}{(x+1)} + \int\left(\frac{1}{x} - \frac{1}{x+1}\right)dx$$

$$= \frac{-\log|x|}{(x+1)} + \log|x| - \log|x+1| + C$$

$$= \frac{-\log|x|}{(x+1)} + \log\left|\frac{x}{x+1}\right| + C$$

24. $|A| = 1(1+6) + 2(2-0) + 0 = 11$

$$\text{adj}\,(A) = \begin{bmatrix} 7 & 2 & -6 \\ -2 & 1 & -3 \\ -4 & 2 & 5 \end{bmatrix}$$

So, $A^{-1} = \frac{1}{|A|}\text{adj}\,(A) = \frac{1}{11}\begin{bmatrix} 7 & 2 & -6 \\ -2 & 1 & -3 \\ -4 & 2 & 5 \end{bmatrix}$

The given system of equations is

$$x - 2y = 10$$
$$2x + y + 3z = 8$$
$$-2y + z = 7$$

The solution of the given system is given by $X = A^{-1}B$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 7 & 2 & -6 \\ -2 & 1 & -3 \\ -4 & 2 & 5 \end{bmatrix}\begin{bmatrix} 10 \\ 8 \\ 7 \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix}$$

$\therefore \quad x = 4, y = -3$ and $z = 1$

Or

Let the three numbers be x, y and z. Then, we have

$$x + y + z = 6;\ x + 2z = 7$$
$$3x + y + z = 12$$

The above equations can be written in matrix form as

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 3 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \\ 12 \end{bmatrix}$$

Now, $|A| = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 3 & 1 & 1 \end{bmatrix} = 1(0-2) - (1-6) + 1(1-0) = 4$

$$\therefore \quad A^{-1} = \frac{1}{4}\begin{bmatrix} -2 & 0 & 2 \\ 5 & -2 & -1 \\ 1 & 2 & -1 \end{bmatrix}$$

Now, $X = A^{-1}B \Rightarrow X = \frac{1}{4}\begin{bmatrix} -2 & 0 & 2 \\ 5 & -2 & -1 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 6 \\ 7 \\ 12 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$

$$\Rightarrow \quad x = 3,\ y = 1 \text{ and } z = 2$$

25. Suppose the factory produces x units of machine A and y units of machine B.

Then, profit

$$z = 10500x + 9000y$$

The mathematical formulation of the problem is

Max $z = 10500x + 9000y$

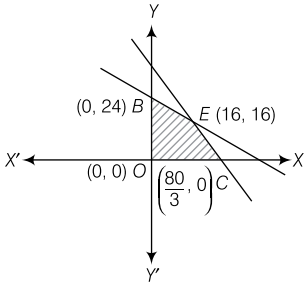
$10x + 20y \leq 480$

$\Rightarrow \quad x + 2y \leq 48$ (metal)

$15x + 10y \leq 400$

$\Rightarrow \quad 3x + 2y \leq 80$ (painting)

$x \geq 0,\ y \geq 0$



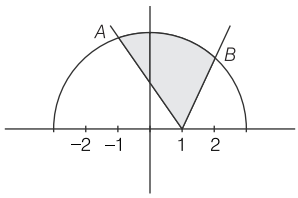
The feasible region is shown in the figure.
The corner points are $O(0, 0)$, $B(0, 24)$, $E(16, 16)$ and $C\left(\frac{80}{3}, 0\right)$.

Corner points	$z = 10500x + 9000y$
(0, 0)	0
(0, 24)	216000
(16, 16)	312000
$\left(\frac{80}{3}, 0\right)$	280000

Thus, profit is maximized by producing 16 units each of machine A and B.

26. Solving $y = \sqrt{5 - x^2}$ and $y = |x - 1|$, we get

$$(x - 1)^2 = 5 - x^2$$
$$\Rightarrow \quad x^2 - x - 2 = 0 \Rightarrow x = 2, -1$$



The required area = The shaded area

$$= \int_{-1}^{1}[\sqrt{5 - x^2} - (1 - x)]\,dx + \int_{1}^{2}[\sqrt{5 - x^2} - (x - 1)]\,dx$$

$$= \int_{-1}^{2}\sqrt{5 - x^2}\,dx - \int_{-1}^{1}(1 - x)\,dx - \int_{1}^{2}(x - 1)\,dx$$

$$= \frac{1}{2}\left[x\sqrt{5 - x^2} + 5\sin^{-1}\left(\frac{x}{\sqrt{5}}\right)\right]_{-1}^{2} - \left[x - \frac{x^2}{2}\right]_{-1}^{1} - \left[\frac{x^2}{2} - x\right]_{1}^{2}$$

$$= \left[-\frac{1}{2} + \frac{5}{5}\left\{\sin^{-1}\left(\frac{2}{\sqrt{5}}\right) + \sin^{-1}\left(\frac{1}{\sqrt{5}}\right)\right\}\right] \text{sq units}$$

27. $I = \int_0^{\pi/2}\frac{x\sin x\cos x}{\sin^4 x + \cos^4 x}dx$

$$I = \int_0^{\pi/2}\frac{(\pi/2 - x)\cos x\sin x}{\cos^4 x + \sin^4 x}dx$$

$$\Rightarrow 2I = \int_0^{\pi/2}\frac{(\pi/2)\cos x\sin x}{\cos^4 x + \sin^4 x}dx$$

$$= \frac{\pi}{2}\left[\int_0^{\pi/4}\frac{\cos x\sin x}{\cos^4 x + \sin^4 x}dx + \int_{\pi/4}^{\pi/2}\frac{\cos x\sin x}{\cos^4 x + \sin^4 x}dx\right]$$

$$= \frac{\pi}{2}\left[\int_0^{\pi/4}\frac{\tan x\sec^2 x}{1 + \tan^4 x}dx + \int_{\pi/4}^{\pi/2}\frac{\text{cosec}^2 x\cot x}{\cot^4 x + 1}dx\right]$$

$$= \frac{\pi}{4}\left[\int_0^1\frac{1}{1 + t^2}dt - \int_1^0\frac{1}{1 + p^2}dp\right]$$

[where, $t = \tan^2 x$ and $p = \cot^2 x$]

$$= \frac{\pi}{4}[\tan^{-1}t]_0^1 + \frac{\pi}{4}[\tan^{-1}p]_0^1 = \frac{\pi^2}{8}$$

$\therefore \quad I = \pi^2/16$

Or

$$I = \int_0^4 (x + e^{2x})dx$$

$$\therefore \int_0^4 (x + e^{2x})dx = \lim_{n\to 0} h[f(0) + f(h) + f(2h) + \ldots + f(n-1)h]$$

$$= \lim_{n\to 0} h[(0 + e^0) + (h + e^{2h}) + (2h + e^{4h}) + \ldots + \{(n-1)h + e^{2(n-1)h}\}]$$

$$= \lim_{n\to 0} h[\{0 + h + 2h + 3h + \ldots + (n-1)h\} + \{1 + e^{2h} + e^{4h} + \ldots + e^{2(n-1)h}\}]$$

$$= \lim_{n\to 0} h\left[\frac{hn(n-1)}{2} + \frac{(e^{2h})^n - 1}{e^{2h} - 1}\right]$$

$$= \lim_{n\to\infty}\frac{16}{n^2} \times \frac{n(n-1)}{2} + \lim_{n\to 0}\frac{\frac{e^{2nh} - 1}{e^{2h} - 1}}{h}$$

$$= \lim_{n\to\infty} 8\left(1 - \frac{1}{h}\right) + \lim_{n\to 0}\frac{e^8 - 1}{\frac{e^{2h} - 1}{2h}}$$

$$= 8(1 - 0) + \frac{e^8 - 1}{2} = \frac{15 + e^8}{2}$$

28. $\mathbf{r} = (2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}}) + \lambda(3\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 2\hat{\mathbf{k}})$

Cartesian equation of line is

$$\frac{x - 2}{3} = \frac{y + 1}{4} = \frac{z - 2}{2} = \lambda \qquad \ldots(i)$$

Coordinates of any point on line are $3\lambda + 2, 4\lambda - 1, 2\lambda + 2$

Equation of plane is $\mathbf{r}\cdot(\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}) = 5$

$$\Rightarrow \quad (x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})\cdot(\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}) = 5$$

$$\Rightarrow \quad x - y + z = 5 \qquad \ldots(ii)$$

As line intersects the plane, so coordinates lies in Eq. (ii),

$\therefore \quad 3\lambda + 2 - 4\lambda + 1 + 2\lambda + 2 = 5 \Rightarrow \lambda = 0$

Thus, coordinates of intersection are $(2, -1, 2)$.

Required distance

$$= \sqrt{(2 + 1)^2 + (-1 + 5)^2 + (2 + 10)^2} = 13$$

Let (x, y, z) be the coordinates of image of point $(-1, -5, -10)$, then

$$\frac{x - 1}{2} = 2, \frac{y - 5}{2} = -1 \text{ and } \frac{z - 10}{2} = 2$$

$$\Rightarrow \quad x = 5, y = 3 \text{ and } z = 14$$

Hence, image of point $(-1, -5, -10)$ in plane is $(5, 3, 14)$.

29. Clearly, $f(x)$ is defined for all $x > 2$.

$$f(x) = 2\log(x - 2) - x^2 + 4x + 1$$

$$\Rightarrow \quad f'(x) = \frac{2}{x - 2} - 2x + 4$$

$$\Rightarrow \quad f'(x) = \frac{-2(x - 1)(x - 3)}{(x - 2)}$$

For $f(x)$ to be increasing, $f'(x) > 0$

$$\Rightarrow \quad \frac{-2(x - 1)(x - 3)}{(x - 2)} > 0$$

$$\Rightarrow \quad (x - 3) < 0$$

$$\Rightarrow \quad x < 3 \Rightarrow x \in (2, 3) \qquad [\because x > 2]$$

So, $f(x)$ is increasing on $(2, 3)$.

For $f(x)$ to be decreasing, $f'(x) < 0$

$$\frac{-2(x - 1)(x - 3)}{(x - 2)} < 0$$

$$\Rightarrow \quad x - 3 > 0 \Rightarrow x > 3$$

$$\Rightarrow \quad x \in (3, \infty)$$

So, $f(x)$ is decreasing on $(3, \infty)$.

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- Question Numbers 13-23 in **Section C** are Long Answers I carrying 4 marks each.
- Question Numbers 24-29 in **Section D** are Long Answers II carrying 6 marks each.

Section A

1. If $R=\{(x, y): x+2y=8\}$ is a relation on N, then write the range of R. [AI 2014]

2. Using the principal value, write the value of $\sec^{-1}(-\sqrt{2})+3\operatorname{cosec}^{-1}(-\sqrt{2})$.

3. If $A=\begin{bmatrix}2 & -3\\ 5 & -6\end{bmatrix}$, then find the value of $|6A|$.

4. What is the cosine of acute angle between $\mathbf{a}$ and $\mathbf{b}$ with magnitudes 2 and 5 respectively when $|\mathbf{a}\times\mathbf{b}|=8$?

Section B

5. Show that the function $f(x)$ defined by

$$f(x)=\begin{cases}\dfrac{1-\cos x}{x^2}, & x\neq 0\\ 1, & x=0\end{cases}$$

is discontinuous at $x=0$.

6. Find the slope of the tangent to the curve $x=a(\theta-\sin\theta)$ and $y=a(1+\cos\theta)$ at $\theta=\dfrac{\pi}{3}$.

7. Evaluate $\int_{\pi/6}^{\pi/3}\dfrac{\sin x-\cos x}{\sin x+\cos x}dx$.

8. If $P(A)=\dfrac{3}{8}$, $P(B)=\dfrac{1}{2}$ and $P(A\cap B)=\dfrac{1}{4}$, then find the value of $P(\overline{A}/\overline{B})$.

9. Find $|\mathbf{a}|$ and $|\mathbf{b}|$, if $(\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}+\mathbf{b})=27$ and $|\mathbf{a}|=2|\mathbf{b}|$.

10. Simplify $\sin^{-1}\left\{\dfrac{\sqrt{1+x}+\sqrt{1-x}}{2}\right\}$, $0<x<1$.

11. Find 2×2 matrix B, such that $AB=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ and $A=\begin{bmatrix}3 & -1\\ -2 & 4\end{bmatrix}$.

12. If $y=\dfrac{x}{x+2}$, then prove that $x\dfrac{dy}{dx}=(1-y)y$.

Section C

13. Using properties of determinants, prove that

$$\begin{vmatrix}a^2+1 & ab & ac\\ ab & b^2+1 & bc\\ ca & cb & c^2+1\end{vmatrix}=1+a^2+b^2+c^2$$

[AI 2011]

14. Discuss the differentiability of the function

$$f(x)=\begin{cases}2x-1, & x<\dfrac{1}{2}\\ 3-6x, & x\geq\dfrac{1}{2}\end{cases}\text{ at } x=\frac{1}{2}.$$

Or

For what value of k is the following function continuous at $x=-\dfrac{\pi}{6}$.

$$f(x)=\begin{cases}\dfrac{\sqrt{3}\sin x+\cos x}{x+\frac{\pi}{6}}, & x\neq-\dfrac{\pi}{6}\\ k, & x=-\dfrac{\pi}{6}\end{cases}$$

15. If $y=e^{m\sin^{-1}x}$, then show that

$$(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}-m^2y=0.$$

[AI 2015]

16. Find the equation of the plane through the point $2\hat{\mathbf{i}}+\hat{\mathbf{j}}-\hat{\mathbf{k}}$ and passing through the line of intersection of the planes $\mathbf{r}\cdot(\hat{\mathbf{i}}+3\hat{\mathbf{j}}-\hat{\mathbf{k}})=0$ and $\mathbf{r}\cdot(\hat{\mathbf{j}}+2\hat{\mathbf{k}})=0$.

Or

Find the shortest distance between the following pairs of lines.

$$\frac{x-1}{-1}=\frac{y+2}{1}=\frac{z-3}{-2}, \frac{x-1}{1}=\frac{y+1}{2}=\frac{z+1}{-2}$$

17. The side of an equilateral triangle is increasing at the rate of 2 cm/s. At what rate is its area increasing when the side of the triangle is 20 cm?

[Delhi 2015]

18. A speaks truth in 60% of the cases and B in 90% of the cases. In what percentage of cases are they likely to contradict each other in stating the same fact?

19. Evaluate $\int_0^{\pi} \frac{x \sin x}{1+\sin x} dx$.

Or

Evaluate $\int_{\pi/4}^{\pi/2} \sqrt{1-\sin 2x}\, dx$.

20. The two vectors $\hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $3\hat{\mathbf{i}} - \hat{\mathbf{j}} + 4\hat{\mathbf{k}}$ represent the two side vectors **AB** and **AC**, respectively of ΔABC. Find the length of the median through A.

[Foreign 2015]

21. Four bad oranges are mixed accidently with 12 good oranges. Find the probability distribution of the number of bad oranges in a draw of two oranges. Also, find the mean of distribution. What are the benefits of oranges in our diet?

22. Evaluate $\int \frac{\log x}{(1+\log x)^2} dx$.

23. Show that $f(x) = \log(1+x) - \frac{2x}{2+x}, x > -1$ is increasing function of x, throughout its domain.

[AI 2012]

Section D

24. A manufacturing company makes two types of products A and B. Each package of product A requires 3 h for assembling and 1 h for testing and each package of product B requires 2 h for assembling and 2 h for testing. For assembling and testing, the maximum labour hours are 15 and 10, respectively. The company makes a profit of ₹ 600 on each package of product A and ₹ 400 on each package of product B. How many package of product A and B should be manufactured per day to realise a maximum profit. Determine the maximum profit.

25. Solve the initial value problem $\sqrt{1-y^2}\, dx = (\sin^{-1} y - x)\, dy$, $y(0) = 0$

Or

Find the particular solution of differential equation.

$x^2 y\, dx - (x^3 + y^3) dy = 0$, for $x = e$ and $y = e$.

26. Using integration, find the area of the ΔABC coordinates of whose vertices are $A(2,3)$, $B(0,1)$ and $C(4,-1)$.

27. Find the equation of the plane determined by the intersection of the lines

$$\frac{x+3}{3}=\frac{y}{-2}=\frac{z-7}{6} \text{ and } \frac{x+6}{1}=\frac{y+5}{-3}=\frac{z-1}{2}$$

Also, find the distance of the point $(2, 3, -5)$ from the obtained plane.

28. Using elementary row transformation, find the inverse of the matrix,

$$A=\begin{bmatrix}1 & 2 & 5\\ 1 & -1 & -1\\ 2 & 3 & -1\end{bmatrix}$$

29. Let * be a binary operation defined on $Q \times Q$ by $(a,b) * (c,d) = (ac, b+ad)$, where Q is the set of rational number. Determine, whether * is commulative and associative. Find the identify element for * and the invertible elements of $Q \times Q$.

Or

Consider, the functions of $f: R \to R$ and $g: R \to R$ defined by $f(x) = 2x - 3$ and $g(x) = \frac{x}{3}$.

Show that f and g are invertible functions and hence verify that $(gof)^{-1} = f^{-1} og^{-1}$.

Answers

1. {1, 2, 3} **2.** 0. **3.** 108

4. $\frac{3}{5}$ **6.** $-\frac{1}{\sqrt{3}}$ **7.** 0

8. $\frac{3}{4}$ **9.** $|\mathbf{a}| = 6, |\mathbf{b}| = 3$ **10.** $\frac{\pi}{4} + \frac{1}{2}\cos^{-1} x$

11. $\frac{1}{10}\begin{bmatrix}4 & 1\\ 2 & 3\end{bmatrix}$ **14.** $k = 2$

16. $\mathbf{r} \cdot (\hat{\mathbf{i}} + 9\hat{\mathbf{j}} + 11\hat{\mathbf{k}}) = 0$ Or $\frac{8}{\sqrt{29}}$

17. $20\sqrt{3}$ cm^2/s **18.** $\frac{42}{100}$ **19.** $\pi\left(\frac{\pi}{2} - 1\right)$ Or $\sqrt{2} - 1$

20. $\frac{\sqrt{34}}{2}$ sq units **21.** Mean $= \frac{1}{2}$ **22.** $\frac{x}{(\log x + 1)} + C$

24. 4 packages of product A and 3 packages of product B. Maximum profit = ₹ 3600

25. $x - \sin^{-1} y + 1 = e^{\sin^{-1} y}$ Or $\log|y| = \frac{2y^3 + x^3}{2y^3}$

26. 6 sq units

27. $2x - z + 13 = 0, \frac{22}{\sqrt{5}}$

28. $\frac{1}{27}\begin{bmatrix}4 & 17 & 3\\ -1 & -11 & 6\\ 5 & 1 & -3\end{bmatrix}$

29. No, yes, $(1,0), \left(\frac{1}{a}, \frac{-b}{a}\right)$

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FUN ARCADE

Augustin-Louis Cauchy

Augustin-Louis Cauchy (21st August, 1789 - 23 May, 1857) was a French mathematician reputed, as a pioneer of analysis. He known for the countless contribution in the domain of Mathematics. He made special contribution to mathematical analysis and the theory of substitution groups. He wrote many book that cover a wide area of mathematics and mathematical physics. Cauchy authored around eight hundred research articles. This paper on definite integral acted as the basic of theory of complex function. Also, his theories on function of complex variables have played substantial part in subjects ranging from applied mathematics to aeronautics. His significant paper on error theory also done valuable role for the domain of science. Cauchy was the first mathematician who developed definition and rules for mathematics. He introduced the definition of the integral and the rule of series convergence. Cauchy remained a profession at the college of de France until his death at the age of 67.
His name is one of the 72 names inscribed on the Eiffel Tower.

"As for methods I have sought to give them all the rigour that one requires in geometry, so as never to have recourse to the reasons drawn from the generality of algebra"

Augustin-Louis Cauchy

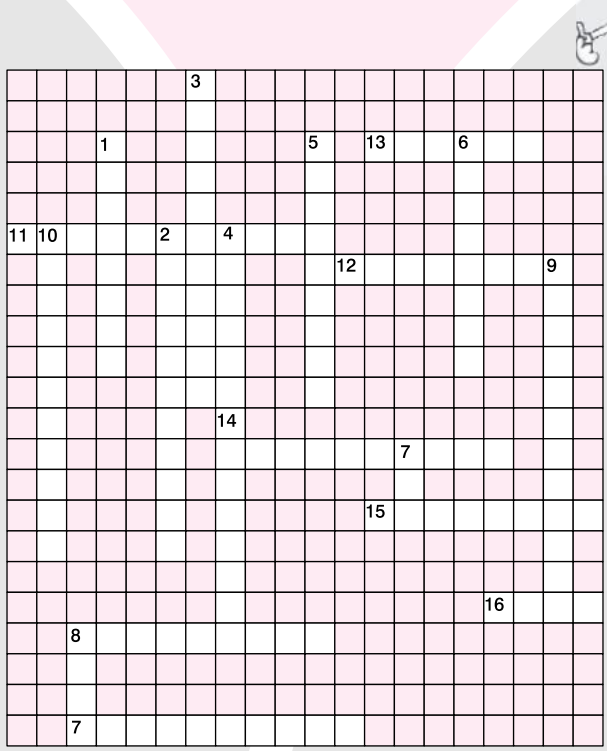
CROSSWORD

DOWN

1. A perpendicular from a vertex of a triangle to the opposite side. (8)
2. If power of y in a equation of curve are all even, then curve is ..... about X-axis. (11)
3. Another name of elementary event. (11)
4. The plane formed by a horizontal axis and a vertical axis is called a ..... (14)
5. The axis which passes through the foci of horizontal ellipse. (9)
6. The order of a differential equation is always a ..... integer. (8)
7. If each element of a row (or column) of a determinant is zero, then its value is ..... . (4)
8. Number of normals drawn from a point on an ellipse. (4)
9. Points that does not lie on a same line. (12)
10. If a triangle is inscribed in a rectangular hyperbola, then its ..... lies on hyperbola. (11)

ACROSS

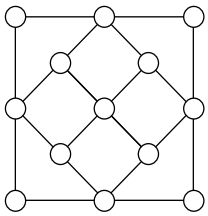
11. The family of curves having circles, ellipse, parabolas and hyperbola. (12)
12. A relation whose each element of domain associated with a unique element of co-domain. (8)
13. A way of multiplying three vectors in which the product is a scalar is called ..... product. (6)
14. A sequence in which the difference of two consecutive term is constant, is called ..... progression. (10)
15. If f and g are continuous functions, then $c(f+g)$ is continuous function, where c is a ..... (8)
16. Vector product of two parallel or collinear vectors is ..... (4)
17. The number of times an observation occurs in the given data is called the ..... of the observation. (9)
18. For the quadratic equation $ax^2+bx+c=0$, if $a=c$, then roots are ..... to each other. (9)



Puzzles to Puzzle You

1. Three brothers shared 24 apples, each getting a number equal to his age 3 yr before.
The youngest one proposed a swap.
"I will keep only half the apples I got, and divide the rest between you two equally. But then the middle brother, keeping half his accumulated apples, must divide the rest equally between the oldest brother and me, and then the oldest brother must do the same".
They agreed, the result was that each ended with apples.
How old were the brothers?

2. We have a crystal lattice whose "atoms" are joined in ten rows of 3 atoms each. Select thirteen integers, of which twelve are different, and place them in the "atoms" so that each row total 20. (The smallest number needed is 1, the largest 15)



3. Two school girls were travelling from the city to a summer cottage on an electric train.
"I notice" one of the girls said that "the trains coming in the opposite direction pass us every 5 min. What do you think how many trains arrive in the city in an hour given equal speed in both direction."
"Twelve of course" the other girl answered, because to divided by 5 equals 12. The first girl did not agree what do you think?

4. A fur dresser had to put a patch shaped like a scalene triangle on a piece of fur.Suddenly, he realized he had made a terrible mistake. The patch fitted the hole but the fur side faced the wrong way. The fur dresser, after some thought, cut the triangular patch into 3 parts, each of which would be unchanged when turned over. How?

5. An engineer goes every day by metro train to the city where he works at 8:30 am as soon as he gets off the train, a car picks him up and takes him to the plant.
One day the engineer takes a train arriving at 7:00 am and start walking toward the plant. On the way, the car picks him up and he arrives at the plant 10 min early.
When does he meet the car?

6. One day a young man and an older man left the village for the city, one on horse, one in a car. Soon it was apparent that if the older man had ridden three times as far as he had, he would have half as far to ride as he had, and if the young man had ridden half as far as he had, he would have three times as far to ride as he had. Who rode the horse?

7. A cubical box contains 27 congruent large balls, its twin contains by congruent smaller balls. All the balls are made of the same material. Both boxes are filled to the top. In each box each layer has the same number of balls and the outside of each layer touch the sides. Which box is heavier?

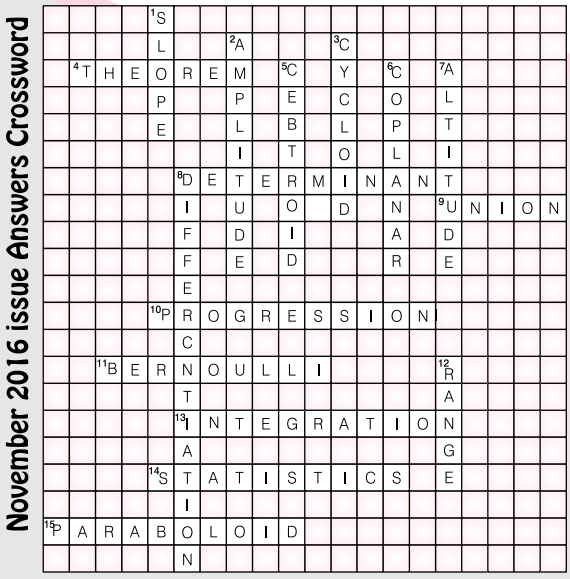
SUDOKU

			2				6	
3						4		1
		1			3		8	
							9	
			5	3	8			
	3							
			3			5		
5		3	7					8
	7				1			

November Answers SUDOKU

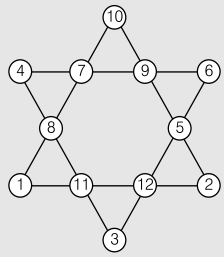
9	6	8	2	1	4	3	7	5
3	2	4	6	7	5	1	8	9
7	5	1	8	3	9	6	4	2
4	7	3	9	6	2	5	1	8
5	8	9	3	4	1	7	2	6
2	1	6	5	8	7	4	9	3
6	4	2	1	9	3	8	5	7
1	3	5	7	2	8	9	6	4
8	9	7	4	5	6	2	3	1

November 2016 issue Answers Crossword



Puzzles to Puzzle You

1. 180 m **2.** 104%
3. 29 eggs
4. From the top clockwise :
10, 1, 11, 7, 4, 6, 12, 2, 8, 5, 9, 3
5.

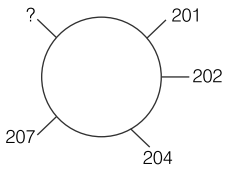


6. 2/6 or 1/3

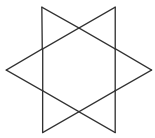
HURRY! WIN ₹1000 IN CASH

JUST SOLVE & SEND KNOWLEDGE COEFFICIENT QUIZZER (NO. 25)

1. $153 = 1^3 + 5^3 + 3^3$
   Find some other three digit numbers like this.
2. Two creepers, one Jasmine and the other rose, are both climbing up and round a cylindrical tree trunk. The Jasmine twists clockwise and the rose anti-clockwise, and they both start at the same point on the ground. Before they reach the first branch on the tree the Jasmine has made 5 complete twists and the rose 3 twists. Not counting the bottom and the top, how many times do they cross?
3. My brother will be x years old in the year x squared. In which year was he born?
4. Guess the number



5. Eighteen sticks form a star with eight triangles (six small ones and two large ones)



   How can just two sticks be moved?
6. Half of zero is still zero. What other number can be halved to make zero?
7. If today is Friday, what is the day that follows the day that comes after the day that precedes the day before yesterday?
8. Which pair of numbers comes next?

42 40	38 35	33 31	28 –

9. What comes next in this sequence
   jnry, fbrry, mrch, prl, my, jn
10. If all the dimensions of a one litre bottle are doubled what is the volume of the new bottle?

KNOWLEDGE Coefficient Quizzer (No. 25)

Write your answers on separate sheets and send to us along with this coupon

(ALL PARTICULARS TO BE FILLED UP IN BLOCK LETTERS)

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Winner of Knowledge Coefficient Quizzer (No. 24) (November Issue)
Raj Kumar (Meerut)

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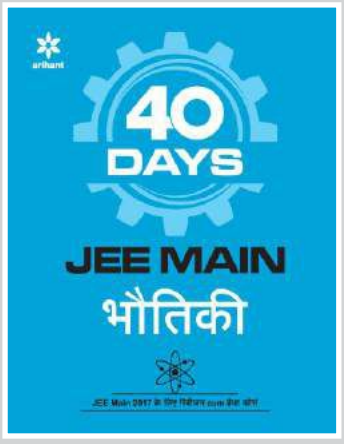

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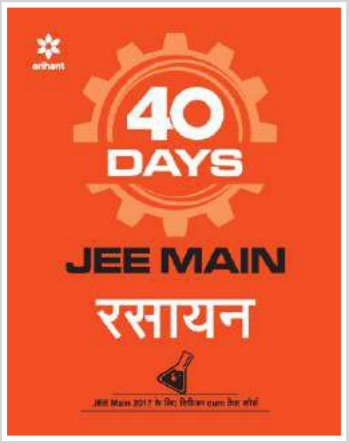

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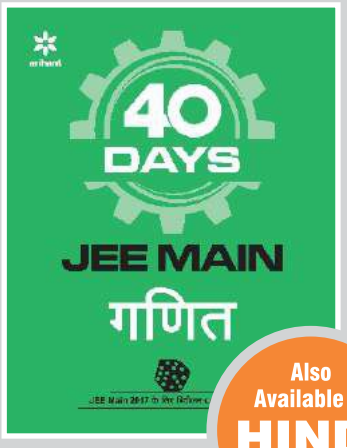

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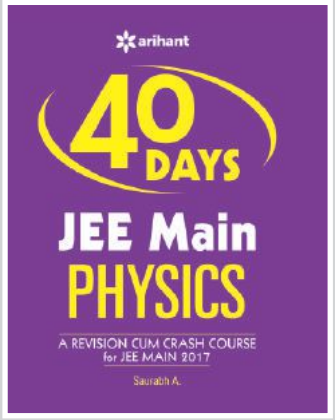

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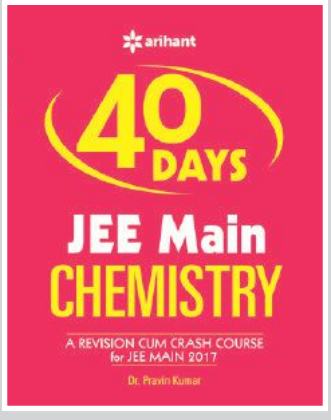

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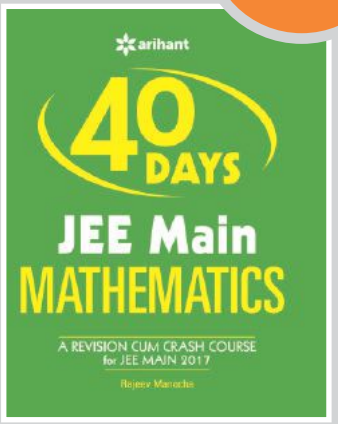

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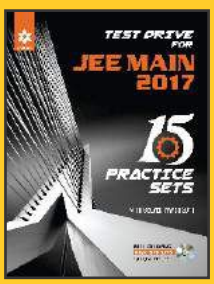

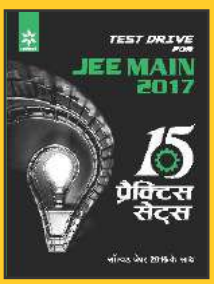

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